# From slightly coloured noises to unitless product systems

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**Abstract**

Stationary Gaussian generalized random processes having slowly decreasing spectral densities give rise to product systems in the sense of William Arveson (basically, continuous tensor product systems of Hilbert spaces). A continuum of nonisomorphic unitless product systems is produced, answering a question of Arveson.

## Introduction

The white noise is a Gaussian stationary generalized random process whose restrictions to adjacent intervals $(a, b)$ and $(b, c)$ are independent. In contrast, for a continuous process, its restrictions to $(a, b)$ and $(b, c)$ are heavily dependent via the value at $b$. Such a dependence cannot be described by a probability density; the joint distribution is singular w.r.t. the product of marginal distributions. By a slightly coloured noise I mean a Gaussian stationary generalized random process such that the distribution of its restriction to $(a, c)$ is absolutely continuous w.r.t. the product of the distributions of its restrictions to $(a, b)$ and $(b, c)$, whenever $-\infty < a < b < c < +\infty$.

In the Hilbert space of all linear functionals of the white noise, every interval $(a, b)$ determines a subspace $G_{a,b}$ satisfying $G_{a,b} \oplus G_{b,c} = G_{a,c}$; the whole space is a direct integral (a continuous direct sum). For arbitrary (not just linear) functionals the relation is multiplicative rather than additive:

$$H_{a,b} \otimes H_{b,c} = H_{a,c} \, ;$$

the whole space is a continuous tensor product, in other words, a product system. The constant 1 may be treated as an element $\mathbf{1}_{a,b} \in H_{a,b}$, satisfying

$$\mathbf{1}_{a,b} \otimes \mathbf{1}_{b,c} = \mathbf{1}_{a,c} \, ;$$

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such a multiplicative family is called a unit (of a product system). There exist product systems with many units, with a single unit (up to a natural equivalence), and unitless (with no unit). However, the theory of unitless product systems suffers from lack of rich sources of examples. Slightly coloured noises are such a source, rich enough for producing a continuum of nonisomorphic unitless product systems.

> "We believe that there should be a natural way of constructing such product systems, and we offer that as a basic unsolved problem. The fact that we do not yet know how to solve it shows how poorly understood continuous tensor products are today."
>
> Arveson 1994 [3, p. 5].

After producing a continuum of nonisomorphic product systems with units [12] I was asked by Arveson (private communication, January 2000) about a continuum of nonisomorphic unitless product systems. His question is answered here by using a construction that was outlined by Tsirelson and Vershik [13, Sect. 1c] with no proofs.

> "This example may be considered as a commutative (bosonic) counterpart of Power's noncommutative (fermionic) example of a non-Fock factorization over $\mathbb{R}$."
>
> Tsirelson and Vershik 1998 [13, p. 91].

## Probabilistic aspects

Probabilistic results are summarized here in probabilistic language (while the rest of the paper is written in rather analytical language).

Consider a Gaussian stationary generalized random process $(\xi_t)_{t\in\mathbb{R}}$ whose covariation function $B(t) = \operatorname{Cov}(\xi_s, \xi_{s+t})$ is positive, decreasing and convex on $(0,\infty)$, and

$$B(t) = \frac{1}{|t| \ln^\alpha(1/|t|)} \quad \text{for all } t \text{ small enough}\,;$$

here $\alpha \in (1,\infty)$ is a parameter.

Then the joint distribution of random variables

$$X_k = \int_0^{2\pi} e^{ikt} \xi_t \, dt \qquad (k = \dots, -2, -1, 0, 1, 2, \dots)$$

has a density (finite and strictly positive almost everywhere) w.r.t. the product of corresponding one-dimensional distributions $N(\mathbb{E}X_k, \operatorname{Var} X_k)$.

The same holds for a larger family $(\dots, X_{-1}, Y_{-1}, X_0, Y_0, X_1, Y_1, X_2, Y_2, \dots)$ of random variables, where $X_k$ are as before, and $Y_k = \int_{-2\pi}^{0} e^{ikt} \xi_t \, dt$.

Another result. Take some $\varepsilon_n \in (0,1)$ and consider random variables

$$Z_n = \frac{1}{\varepsilon_n} \sum_{k=0}^{n-1} \int_{k/n}^{(k+\varepsilon_n)/n} \xi_t \, dt \,, \qquad Z = \int_0^1 \xi_t \, dt \,.$$

If $\varepsilon_n \ln^{\alpha-1} n \to \infty$ then $Z_n \to Z$ in $L_2$.

If $\varepsilon_n \ln^{\alpha-1} n \to 0$ then $\operatorname{Var}(Z_n) \to \infty$ and the correlation coefficient $\operatorname{Corr}(Z_n, Z) \to 0$.

If $\lim_n (\varepsilon_n \ln^{\alpha-1} n) \in (0,\infty)$ then $\lim_n \operatorname{Var}(Z_n) \in (0,\infty)$ and $\lim_n \operatorname{Corr}(Z_n, Z) \in (0,1)$.

For detail see Sect. 10. The reader interested just in these probabilistic statements may skip sections 1–9 in which case, however, he/she should bypass some points of analytical nature in Sect. 10, and restore some proofs omitted since they are not necessary from the analytical viewpoint.

## Quantal aspects

Acquaintance with quantum theory is not needed for reading the rest of the paper, but should help to understand the idea as explained here.

A product system may be thought of as a local quantum field over the one-dimensional space $\mathbb{R}$ (just space, no time at all; see also Arveson [2] for a better, dynamical interpretation of product systems). The whole field is a quantum system, and its restriction to $(0,1)$ is a subsystem. A unit vector of $H_{0,1}$ describes a pure state of the subsystem (though in general the subsystem and the rest of the system are entangled).

Local Hilbert spaces (and algebras) are ascribed to intervals, as well as to more general regions, consisting of a finite number of intervals. Introduce a region $E_n$ consisting of $n$ small equidistant intervals of equal length,

$$E_n = \Big( 0, \frac{\varepsilon_n}{n} \Big) \cup \Big( \frac{1}{n}, \frac{1+\varepsilon_n}{n} \Big) \cup \dots \cup \Big( \frac{n-1}{n}, \frac{n-1+\varepsilon_n}{n} \Big)$$

and consider the corresponding subsystem, described by its Hilbert space $H_{E_n}$. For a fixed $n$, the subsystem may be entangled or not (with the rest). If the product system has a unit then there is a 'white state' with no spatial correlations. It makes $H_{E_n}$ disentangled for all $n$ simultaneously. That is the case for a product system constructed out of the white noise.

A unitless product system is constructed out of the slightly coloured noise mentioned in 'Probabilistic aspects'. Spatial correlations inherent to the

noise are inherited by quantum states. Subsystems $H_{E_n}$ can be disentangled for finitely many $n$, but not for all $n$ simultaneously.

Asymptotic behavior of subsystems $H_{E_n}$ depends crucially on $\lim(\varepsilon_n \ln^{\alpha-1} n)$, as is suggested by properties of random variables $Z_n$ (see 'Probabilistic aspects'). If $\lim(\varepsilon_n \ln^{\alpha-1} n) = 0$ then $E_n$ are a tail sequence in the sense that the mixed state of the subsystem (the density matrix on $H_{E_n}$) has a universal asymptotics, irrespective of the state of the system. The situation is different if $\lim(\varepsilon_n \ln^{\alpha-1} n) \neq 0$. This is why product systems for different $\alpha$ are nonisomorphic.

## Aspects of functional analysis

A classical construction going back to Fock may be outlined as follows. One starts with a Hilbert space $G$. One identifies $G$ with the space of all measurable linear functionals over a Gaussian measure $\gamma$. One gets another Hilbert space $H = L_2(\gamma)$ that may be denoted $H = \operatorname{Exp} G$, since $G = G_1 \oplus G_2$ implies $H = H_1 \otimes H_2$ (via $\gamma = \gamma_1 \otimes \gamma_2$). A continuous direct sum, $G = \int^{\oplus} G_\zeta \, d\zeta$ (basically the same as a projection-valued measure well-known in spectral theory) leads to a continuous tensor product. Classical product systems, obtained this way, contain units.

My modification of the classical construction is rather innocent (but surprisingly powerful). Basically, the subspaces $G_1, G_2$ are allowed to be slightly nonorthogonal, without destroying the relation $H = H_1 \otimes H_2$. In terms of Gaussian measures, the relation $\gamma = \gamma_1 \otimes \gamma_2$ is generalized to $\gamma = p \cdot (\gamma_1 \otimes \gamma_2)$ where $p$ is a density (that is, Radon-Nikodym derivative). The density $p$ is inserted properly into the formula for $h_1 \otimes h_2 \in H$.

The classical language (of Hilbert spaces, Gaussian measures, spaces $L_2(\gamma)$ etc.) is not well-suited to the modified construction. A modified language, used in the paper, stipulates larger invariance (symmetry) groups. Namely, the 'Hilbert space' structure corresponds to the group of all unitary operators, $U^*U = I$. I use a larger group consisting of all invertible $U$ such that $U^*U - I$ is a Hilbert-Schmidt operator. (These $U$ are called 'equivalence operators' in Feldman's well-known paper [7] on equivalence of Gaussian measures). The corresponding structure, weaker than 'Hilbert space' structure but stronger than 'linear topological space' structure, is defined in Sect. 2 under the name 'FHS-space'. An equivalence class (in Feldman's sense) of norms is used rather than a single norm.

Accordingly, a measure space $(\Omega, \mathcal{F}, P)$ is replaced with a 'measure type space' $(\Omega, \mathcal{F}, \mathcal{P})$. An equivalence class $\mathcal{P}$ of measures is used rather than a single measure $P$. See Sect. 1. It appears that $L_2(\Omega, \mathcal{F}, \mathcal{P})$ can be defined naturally (in addition to the usual $L_2(\Omega, \mathcal{F}, P)$); see Sect. 1.

Also, a 'Gaussian type space' defined in Sect. 2 stipulates an equivalence class of Gaussian measures rather than a single Gaussian measure.

For a technical reason we need Borel measurability of several natural constructions. For example, the orthogonal projection of a vector to a subspace (in a Hilbert space) is a jointly Borel measurable function of the point and the subspace, provided that the set of all subspaces is equipped with its natural Borel structure. Similarly, the conditional expectation is a jointly Borel measurable function of a random variable and a sub-$\sigma$-field. See Sect. 6 for detail.

# 1 Measure type spaces and square roots of measures

Let $\Omega$ be a nonempty set, $\mathcal{F}$ a $\sigma$-field of its subsets, and $\mu, \nu$ measures[2] (real-valued, finite, positive) on $\mathcal{F}$. One says that $\mu, \nu$ are *equivalent,* if they are mutually absolutely continuous. Let $\mathcal{P}$ be an equivalence class of probability measures on $(\Omega, \mathcal{F})$. That is, every $P \in \mathcal{P}$ is a measure on $(\Omega, \mathcal{F})$ satisfying $P(\Omega) = 1$, and every $P_1, P_2 \in \mathcal{P}$ are equivalent, and $\mathcal{P}$ contains every probability measure equivalent to a measure of $\mathcal{P}$ (and of course, $\mathcal{P}$ is nonempty). One says that $\mathcal{P}$ is a type of measure, and $(\Omega, \mathcal{F}, \mathcal{P})$ is a *measure type space.*

The linear topological (metrizable, but not locally convex) space $L_0(\Omega, \mathcal{F}, \mathcal{P})$ consists of all (equivalence classes of) measurable functions on $\Omega$. Its topology corresponds to convergence in measure (in probability), irrespective of the choice of a measure $P \in \mathcal{P}$.

Hilbert spaces $L_2(\Omega, \mathcal{F}, P_1)$ and $L_2(\Omega, \mathcal{F}, P_2)$ for $P_1, P_2 \in \mathcal{P}$ differ (unless $\exists \varepsilon \;\; \varepsilon P_1 \le P_2 \le (1/\varepsilon) P_1$). However, they are in a *natural* unitary correspondence. Namely, $\psi_1 \in L_2(\Omega, \mathcal{F}, P_1)$ corresponds to $\psi_2 \in L_2(\Omega, \mathcal{F}, P_2)$ when $\psi_2 = \sqrt{\frac{P_1}{P_2}} \psi_1$; here and henceforth $\frac{P_1}{P_2}$ stands for the Radon-Nikodym density, and I write $P_1 = \frac{P_1}{P_2} \cdot P_2$. We'll glue all $L_2(\Omega, \mathcal{F}, P)$ together, forming $L_2(\Omega, \mathcal{F}, \mathcal{P})$ that contains $\sqrt{P}$ for all $P \in \mathcal{P}$, as we'll see soon.

There are several reasons for introducing 'square roots of measures'. One

[2] I assume always, that every measure space $(\Omega, \mathcal{F}, \mu)$ is a Lebesgue-Rokhlin space (that is, isomorphic mod0 to an interval with Lebesgue measure, to a finite or countable set of atoms, or a combination of both). However, the assumption is not really used in this work. We deal with such objects as $L_2(\Omega, \mathcal{F}, \mu)$; the latter must be separable; other properties of $(\Omega, \mathcal{F}, \mu)$ do not matter. I assume also that all considered $\sigma$-fields ($\mathcal{F}$ itself, and its sub-$\sigma$-fields) contain all negligible sets. Everything is treated mod0, that is, up to negligible sets.

reason. In quantum mechanics, Schrödinger's wave function $\psi(x)$ determines a probability distribution $|\psi(x)|^2\,dx$. However, for infinitely many degrees of freedom we have no Lebesgue measure $(dx)$. It could be convenient to describe a quantum state by an object that combines a measure and phases, something like $\psi(x)\sqrt{dx}$.

Old-fashioned tensor analysis stipulates a notion of a relative tensor, in particular, a relative scalar of a given weight (see for instance [6, item 156 on pp. 345–346]). A density of a measure on a manifold is a relative scalar field of weight 1. Relative scalar fields of weight $1/2$ are smooth finite-dimensional 'square roots of measures' considered below.

Define $L_2(\Omega,\mathcal{F},\mathcal{P})$, denoted also by $L_2(\mathcal{P})$ for short, as the Hilbert space of all families $\psi=(\psi_P)_{P\in\mathcal{P}}$ such that $\psi_P\in L_2(\Omega,\mathcal{F},P)$ for every $P\in\mathcal{P}$, and

$$\psi_{P_2}=\sqrt{\frac{P_1}{P_2}}\psi_{P_1} \tag{1.1}$$

for all $P_1,P_2\in\mathcal{P}$. Linear operations and scalar product are defined naturally; for every $P\in\mathcal{P}$, the map $L_2(\mathcal{P})\ni\psi\mapsto\psi_P\in L_2(P)$ is unitary.

Given $P\in\mathcal{P}$, we define $\sqrt{P}\in L_2(\mathcal{P})$ by $(\sqrt{P})_P=1$ (identically on $\Omega$). More generally, given $P\in\mathcal{P}$ and $f\in L_2(P)$, we define $f\sqrt{P}\in L_2(\mathcal{P})$ by $(f\sqrt{P})_P=f$. Thus, $\psi=\psi_P\sqrt{P}$ for all $P\in\mathcal{P}$, $\psi\in L_2(\mathcal{P})$. Now we may replace the notation $\psi_P$ by a more expressive notation

$$\psi_P=\frac{\psi}{\sqrt{P}}\,.$$

Given $\psi',\psi''\in L_2(\mathcal{P})$, their scalar product is $\langle\psi',\psi''\rangle=\int\frac{\psi'}{\sqrt{P}}\frac{\psi''}{\sqrt{P}}\,dP$; the latter does not depend on $P\in\mathcal{P}$. Heuristically we could write $\langle\psi',\psi''\rangle=\int\psi'(\omega)\psi''(\omega)$, however, I prefer the notation

$$\langle\psi',\psi''\rangle=\int\frac{\psi'(\omega)}{\sqrt{P(d\omega)}}\frac{\psi''(\omega)}{\sqrt{P(d\omega)}}\,P(d\omega)=\int\psi'(\omega)\psi''(\omega)\frac{d\omega}{d\omega}\,;$$

being rather awkward and illogical,[3] it still helps. Given $\psi',\psi''\in L_2(\mathcal{P})$, we define the signed measure $\psi'\psi''$ by $\frac{\psi'\psi''}{P}=\frac{\psi'}{\sqrt{P}}\frac{\psi''}{\sqrt{P}}$, thus $\langle\psi',\psi''\rangle=(\psi'\psi'')(\Omega)$, the measure of the whole space. Note also that $\langle\sqrt{P_1},\sqrt{P_2}\rangle=\int\sqrt{P_1(d\omega)}\sqrt{P_2(d\omega)}$ was used in Kakutani's well-known work [9] about equivalence of product measures. Two natural metrics on $\mathcal{P}$ define the same topology (the only one used here); I mean the variation distance, $\|P_1-P_2\|=\int\left|\frac{P_1}{P}-\frac{P_2}{P}\right|dP$, and the angle in $L_2(\mathcal{P})$, $\arccos\langle\sqrt{P_1},\sqrt{P_2}\rangle$; the proof is left to the reader.[4]

[3]Neither $\psi(\omega)$ nor $\psi(d\omega)$ is a logical notation for an object of the form $f(\omega)\sqrt{P(d\omega)}$.

[4]In fact, $\left(\frac12\|P_1-P_2\|\right)^2\le1-\langle\sqrt{P_1},\sqrt{P_2}\rangle\le\frac12\|P_1-P_2\|$, which is not used here.

## 2 Gaussian measures, quadratic norms, FHS

One may introduce a Gaussian measure as a probability measure on a Banach (or Hilbert, or Frechet, etc) space, such that every (continuous) linear functional has a normal distribution. Such a viewpoint is convenient for heuristic thinking but not for a formal presentation, since topological structures on the linear space with measure are in fact irrelevant. Following an old advice of A. Vershik, I prefer discarding any topological structure on the (Banach, etc.) space and even keeping implicit its linear structure. Everything can be formulated in terms of two different spaces, a probability space and a Hilbert space; the latter plays the role of the space of all measurable linear functionals, as well as its dual, the space of all admissible shifts.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $G \subset L_2(\Omega, \mathcal{F}, P)$ a (closed) linear subspace, containing constants and generating the whole $\sigma$-field $\mathcal{F}$. We call $G$ a *Gaussian space,* if every $g \in G$ has a normal distribution; the latter is therefore $N(\mathbb{E}g, \operatorname{Var} g) = N\big(\int g\,dP, \int g^2\,dP - (\int g\,dP)^2\big)$. Up to isomorphism, there is exactly one Gaussian space in every dimension $(0, 1, 2, \dots$ or $\infty)$. That is, if $G_1 \subset L_2(\Omega_1, \mathcal{F}_1, P_1)$ and $G_2 \subset L_2(\Omega_2, \mathcal{F}_2, P_2)$ are Gaussian spaces and $\dim G_1 = \dim G_2$, then there exists an isomorphism (invertible measure preserving map) between the probability spaces $(\Omega_1, \mathcal{F}_1, P_1)$ and $(\Omega_2, \mathcal{F}_2, P_2)$ that induces an isometry betwen $G_1$ and $G_2$. Moreover, each isometry between $G_1$ and $G_2$ corresponds to exactly one isomorphism of probability spaces.

The standard model is the space $\mathbb{R}^\infty = \mathbb{R} \times \mathbb{R} \times \dots$ of all sequences of real numbers, equipped with the product measure $\gamma^\infty = \gamma^1 \otimes \gamma^1 \otimes \dots$ where $\gamma^1$ is the standard normal distribution $N(0,1)$. Here $G$ consists of all measurable linear functionals $\mathbb{R}^\infty \ni (x_1, x_2, \dots) \mapsto \sum c_k x_k \in \mathbb{R}$ with $\sum c_k^2 < \infty$; that is, $l_2$ is the standard model of $G$. Any other model is necessarily isomorphic to the standard model, as far as $G$ is of infinite dimension; straightforward modifications for a finite dimension are left to the reader.

The same $G$ over $\mathbb{R}^\infty$ is also a Gaussian space w.r.t. many other measures on $\mathbb{R}^\infty$ equivalent to the standard measure $\gamma^\infty$. In particular, consider the product measure $\gamma$ on $\mathbb{R}^\infty$ whose $k$-th factor is the normal distribution $N(m_k, \sigma_k^2)$ with given parameters $m_k \in (-\infty, +\infty)$, $\sigma_k \in (0, +\infty)$. The well-known S. Kakutani's theorem on equivalence of infinite product measures [9] shows that $\gamma$ is equivalent to $\gamma^\infty$ if and only if

$$\sum_k (\sigma_k - 1)^2 < \infty \quad \text{and} \quad \sum_k m_k^2 < \infty \,.$$

Replacing coordinate axes by arbitrary orthogonal (in $l_2$) axes one gets all Gaussian measures on $\mathbb{R}^\infty$ equivalent to $\gamma^\infty$, which is basically the well-known

criterion (Feldman [7], Hajek [8], Segal [10]).

In terms of a Gaussian space $G \subset L_2(\Omega, \mathcal{F}, P)$ we introduce the set $\mathcal{P}$ of all probability measures on $(\Omega, \mathcal{F})$ equivalent to $P$, and its subset $\mathcal{P}_G$ consisting of all $\gamma \in \mathcal{P}$ such that $G$ is also a Gaussian space in $L_2(\Omega, \mathcal{F}, \gamma)$; these measures will be called *Gaussian measures w.r.t. $G$.* Now we forget the initial measure $P$; each measure of $\mathcal{P}_G$ may serve as $P$ equally well. Up to isomorphism, the structure $(\Omega, \mathcal{F}, \mathcal{P}, G)$ is uniquely determined by $\dim G$. I call $(\Omega, \mathcal{F}, \mathcal{P}, G)$ a *Gaussian type space.*

We introduce a quotient space $G_0 = G/\operatorname{Const}$ (where Const is the one-dimensional space of constants) and its dual space $G^0$ (consisting of all continuous linear functionals on $G_0$). Every $\gamma \in \Gamma$ determines a Hilbert norm $\| \cdot \|_\gamma$ on $G_0$ (that is, $(G_0, \| \cdot \|_\gamma)$ is a Hilbert space) via

$$\|g\|_\gamma^2 = \operatorname{Var}_\gamma(g) = \int g^2 \, d\gamma - \Big( \int g \, d\gamma \Big)^2 ,$$

which is insensitive to any constant added to $g \in G$. Each norm $\| \cdot \|_\gamma$ defines the same topology on $G_0$. However, the norms are equivalent in a stronger sense, namely, for any $\gamma_1, \gamma_2 \in \Gamma$ there exists a basis $(g_k)$ of $G_0$, orthogonal for both norms and such that the numbers $\lambda_k = \|g_k\|_{\gamma_2} / \|g_k\|_{\gamma_1}$ satisfy $\lambda_k > 0$ and $\sum (\lambda_k - 1)^2 < \infty$ (which can be reformulated in terms of Hilbert-Schmidt operators; namely, the unit operator $(G_0, \| \cdot \|_{\gamma_1}) \ni g \mapsto g \in (G_0, \| \cdot \|_{\gamma_2})$ must be an equivalence operator, as defined by Feldman [7, Def. 1]). I'll call such norms *FHS-equivalent.*[5] Of course, it is an equivalence relation [7, Lemma 2]. Numbers $\lambda_k$ determine the distance between $\gamma_1, \gamma_2$ (provided that $\gamma_1, \gamma_2$ have the same mean, that is, $\int g \, d\gamma_1 = \int g \, d\gamma_2$ for all $g \in G$); namely, a simple (basically, one-dimensional) calculation gives

$$\langle \sqrt{\gamma_1}, \sqrt{\gamma_2} \rangle = \prod_k \left( \frac{\lambda_k^{-1/2} + \lambda_k^{1/2}}{2} \right)^{-1/2} ;$$

of course, convergence of the product is equivalent to convergence of the series $\sum_k (\lambda_k - 1)^2$.

**2.1. Definition.** An *FHS-space* is a pair $(H, \mathcal{N})$ of a linear space $H$ and a set $\mathcal{N}$ of norms on $H$ such that

(a) every norm of $\mathcal{N}$ turns $H$ into a separable Hilbert space;

(b) all norms of $\mathcal{N}$ are pairwise FHS-equivalent;

(c) every norm FHS-equivalent to a norm of $\mathcal{N}$ belongs to $\mathcal{N}$.

Norms belonging to $\mathcal{N}$ will be called *admissible norms* on $H$.

[5] You may interpret FHS as Feldman-Hajek-Segal, or alternatively as Feldman-Hilbert-Schmidt.

Isomorphisms of FHS-spaces are basically the same as Feldman's equivalence operators [7, Def. 1]. Up to isomorphism, there is exactly one FHS-space in every dimension ($0, 1, 2, \dots$ or $\infty$); just the same situation as for (separable) Hilbert spaces.

Every separable Hilbert space is isometric to a Gaussian space (over some probability space), and this superstructure brings no arbitrariness, as far as it is considered up to isomorphism. (One speaks about *the* isonormal random process on any given Hilbert space.) Similarly, every FHS-space may be identified with $G/\operatorname{Const}$ for *the* Gaussian type space $(\Omega, \mathcal{F}, \mathcal{P}, G)$ of the corresponding dimension. Thus, in principle one may prove a purely geometric statements about an FHS-space via Gaussian measures, and this way is indeed used in the next section. The correspondence between $\mathcal{N}$ (admissible norms) and $\mathcal{P}_G$ (Gaussian measures) transfers to $\mathcal{N}$ the natural topology of $\mathcal{P}_G$ (inherited from $\mathcal{P}$; recall the end of Sect. 1). In terms of the numbers $\lambda_k$, a basis of neighborhoods of a given admissible norm may be written as $\sum_k (\lambda_k - 1)^2 < \varepsilon$.

The (additive group of) space $G^0$, dual to $G_0$, acts on $(\Omega, \mathcal{F}, \mathcal{P})$ by automorphisms (invertible measurable transformations preserving $\mathcal{P}$) $U_x$, $x \in G^0$, such that

$$g(U_x \omega) - g(\omega) = \langle g, x \rangle \quad \text{for almost all } \omega \in \Omega$$

for all $g \in G$. Clearly, $U_x$ is uniquely determined. Existence of these $U_x$ may be checked just for the standard model, which makes it evident: $G^0 = l_2$ acts on $\mathbb{R}^\infty$ by shifts, $U_x(\omega) = \omega + x$. The map $U_x$ sends each Gaussian measure $\gamma_1 \in \mathcal{P}_G$ into another Gaussian measure $\gamma_2 \in \mathcal{P}_G$ such that $\| \cdot \|_{\gamma_1} = \| \cdot \|_{\gamma_2}$ on $G_0$, and $\int g \, d\gamma_2 - \int g \, d\gamma_1 = \langle g, x \rangle$ for all $g \in G$. A simple (basically, one-dimensional) calculation gives

$$\langle \sqrt{\gamma_1}, \sqrt{\gamma_2} \rangle = \exp\bigl( -\tfrac{1}{8} \|x\|^2 \bigr) \, ,$$

where $\| \cdot \| = \| \cdot \|_{\gamma_1} = \| \cdot \|_{\gamma_2}$. In fact, $\sqrt{\gamma_1} \cdot \sqrt{\gamma_2} = \exp\bigl( -\frac{1}{8} \|x\|^2 \bigr) \gamma$, where $\gamma$ is the image of $\gamma_1$ under $U_{x/2}$.

Turn to the Hilbert space $L_2(\Omega, \mathcal{F}, \mathcal{P})$ of 'square roots of measures'. Transformations $U_x$ induce unitary operators on $L_2(\mathcal{P})$; I denote them by $U_x$, too. Namely, for every $\psi \in L_2(\mathcal{P})$ and $P \in \mathcal{P}$,

$$\frac{\psi}{\sqrt{P}}(\omega) = \frac{U_x \psi}{\sqrt{U_x P}}(U_x \omega) \, ,$$

where $U_x P$ (denoted also by $P \circ U_x^{-1}$) is the image of $P$ under $U_x$. Thus, the FHS-space $G^0$ acts unitarily on the Hilbert space $L_2(\mathcal{P})$.

There is also a natural projective action of the other FHS-space, $G_0$, on $L_2(\mathcal{P})$. Namely, every $g \in G$ determines a unitary operator $V_g : L_2(\mathcal{P}) \to L_2(\mathcal{P})$,

$$V_g \psi = e^{ig} \psi \,,$$

the multiplication by the function $\omega \mapsto e^{ig(\omega)}$. Given $y \in G_0$, we get $V_y$ determined up to a phase factor, which means a *projective* action. In fact, one can choose phase factors getting a unitary representation (which is evident for the standard model). Anyway,

$$V_y U_x = e^{i\langle x,y \rangle} U_x V_y \,,$$

the well-known Weyl form of Canonical Commutation Relations. In this context, vectors $\psi \in L_2(\mathcal{P})$ of the form $\psi = V_y \sqrt{\gamma}$, $\gamma \in \Gamma$, $y \in G_0$, are known as coherent states, or quasi-free pure states, or Gaussian pure states. It is easy to see that

$$\langle V_y \sqrt{\gamma}, \sqrt{\gamma} \rangle = \exp\bigl( -\tfrac12 \|y\|_\gamma^2 \bigr)$$

for $y \in G_0$.

# 3 Some geometry via measure theory

Many statements of this section are purely geometric and probably could be proved within the Hilbert space geometry, but I find it easier to prove them via measure theory. Usually, the corresponding properties of Gaussian measures hold for arbitrary (non-Gaussian) measures as well.

The formula $H = H_1 \oplus H_2$ has several interpretations. It always implies that $H_1, H_2$ are subspaces of $H$ and every vector $h \in H$ has a unique representation as $h_1 + h_2$ where $h_1 \in H_1$, $h_2 \in H_2$. However, when $H$ is a Hilbert space, one usually stipulates that $H_1, H_2$ are orthogonal. When $H$ is an FHS-space, we treat

$$H = H_1 \oplus H_2$$

as follows: there exists an *admissible*[6] norm on $H$ that makes $H_1, H_2$ orthogonal (and of course, they span the whole $H$). Similarly, $H = H_1 \oplus \dots \oplus H_n$ means existence of an admissible norm that makes $H_1, \dots, H_n$ orthogonal (also, they span $H$).

[6] Recall Definition 2.1.

**3.1. Proposition.** Let $H$ be an FHS-space, and $H_1, H_{12}, H_{23}, H_3$ its subspaces such that $H_1 \subset H_{12}$, $H_{23} \supset H_3$, and

$$H_{12} \oplus H_3 = H = H_1 \oplus H_{23} \,.$$

Then the subspace $H_2 = H_{12} \cap H_{23}$ satisfies

$$H_1 \oplus H_2 \oplus H_3 = H \,, H_1 \oplus H_2 = H_{12} \,, H_2 \oplus H_3 = H_{23} \,.$$

Note. Given that $H_1 \oplus H_2 \oplus H_3 = H$, other relations $H_1 \oplus H_2 = H_{12}$, $H_2 \oplus H_3 = H_{23}$ may be treated in the topological sense.

*Proof.* The decomposition $H = H_1 \oplus H_{23}$ determines a projection $P_1 : H \to H$ such that $P_1 H = H_1$ and $1 - P_1 = P_{23}$ is also a projection, $P_{23} H = H_{23}$. The same for $P_3$ and $P_{12} = 1 - P_3$. The inclusion $H_1 \subset H_{12}$ gives $P_1 = P_{12} P_1 = (1 - P_3) P_1$, that is, $P_3 P_1 = 0$; similarly, $P_1 P_3 = 0$. So, our projections commute with each other. Introducing

$$P_2 = P_{12} P_{23} = P_{23} P_{12} = (1 - P_1)(1 - P_3) = 1 - P_1 - P_3 \,,$$

we have $P_2^2 = P_{12} P_{23} P_{12} P_{23} = P_{12}^2 P_{23}^2 = P_{12} P_{23} = P_2$; that is, $P_2$ is also a projection, and $P_1 + P_2 + P_3 = 1$. It follows that $P_2 H = \big( (P_1 + P_2) H \big) \cap \big( (P_2 + P_3) H \big) = H_{12} \cap H_{23} = H_2$. So, the relation $H = H_1 \oplus H_2 \oplus H_3$ holds in the topological sense (that is, when $H$ is treated as a linear topological space). The following lemma completes the proof. □

**3.2. Lemma.** Let $H$ be an FHS-space, $H_1, H_2, H_3$ its subspaces such that $H = H_1 \oplus H_2 \oplus H_3$ in the topological sense, and $(H_1 + H_2) \oplus H_3 = H = H_1 \oplus (H_2 + H_3)$ in the FHS sense (the bracketed sums being topological). Then $H = H_1 \oplus H_2 \oplus H_3$ in the FHS sense.

*Proof.* We introduce a Gaussian type space $(\Omega, \mathcal{F}, \mathcal{P}, G)$ and identify $H$ with $G_0$. Subspaces $H_1, H_2, H_3$ generate sub-$\sigma$-fields $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3 \subset \mathcal{F}$. Note that $H_1 + H_2$ generates $\mathcal{F}_1 \vee \mathcal{F}_2$, the least $\sigma$-field containing both $\mathcal{F}_1$ and $\mathcal{F}_2$. The following two lemmas (and one definition) complete the proof. □

**3.3. Definition.** Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a measure type space, and $\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_n \subset \mathcal{F}$ sub-$\sigma$-fields. We write $\mathcal{F}_0 = \mathcal{F}_1 \otimes \dots \otimes \mathcal{F}_n$, if $\mathcal{F}_1 \vee \dots \vee \mathcal{F}_n = \mathcal{F}_0$ and there exists $P \in \mathcal{P}$ making $\mathcal{F}_1, \dots, \mathcal{F}_n$ independent.[7]

[7]Which means $P(A_1 \cap \dots \cap A_n) = P(A_1) \dots P(A_n)$ for all $A_1 \in \mathcal{F}_1, \dots, A_n \in \mathcal{F}_n$.

**3.4. Lemma.** Let $(\Omega, \mathcal{F}, \mathcal{P}, G)$ be a Gaussian type space, $H = G_0$ the corresponding FHS-space, $H_1, \dots, H_n \subset H$ subspaces, and $\mathcal{F}_1, \dots, \mathcal{F}_n \subset \mathcal{F}$ corresponding sub-$\sigma$-fields. Then

$$H = H_1 \oplus \dots \oplus H_n \quad \text{if and only if} \quad \mathcal{F} = \mathcal{F}_1 \otimes \dots \otimes \mathcal{F}_n \, .$$

*Proof.* 'Only if': take a Gaussian measure $\gamma \in \Gamma$ such that $\| \cdot \|_\gamma$ makes $H_1, \dots, H_n$ orthogonal, then $\gamma$ makes $\mathcal{F}_1, \dots, \mathcal{F}_n$ independent.

'If': some $P \in \mathcal{P}$ makes $\mathcal{F}_1, \dots, \mathcal{F}_n$ independent; however, $P$ need not be Gaussian. We take any Gaussian measure $\gamma_0 \in \Gamma$ and introduce $\mathcal{F}_k$-measurable densities

$$f_k = \frac{\gamma_0|_{\mathcal{F}_k}}{P|_{\mathcal{F}_k}} \quad \text{for } k = 1, \dots, n \, .$$

The measure $\gamma = f_1 \dots f_n \cdot P$ still makes $\mathcal{F}_1, \dots, \mathcal{F}_n$ independent, and $\gamma|_{\mathcal{F}_k} = f_k \cdot (P|_{\mathcal{F}_k}) = \gamma_0|_{\mathcal{F}_k}$. So, w.r.t. $\gamma$ the spaces $H_1, \dots, H_n$ are *independent* Gaussian spaces. Therefore their sum $H$ is Gaussian w.r.t. $\gamma$, that is, $\gamma \in \Gamma$. □

**3.5. Lemma.** Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a measure type space, and $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3 \subset \mathcal{F}$ sub-$\sigma$-fields such that

$$(\mathcal{F}_1 \vee \mathcal{F}_2) \otimes \mathcal{F}_3 = \mathcal{F} = \mathcal{F}_1 \otimes (\mathcal{F}_2 \vee \mathcal{F}_3) \, .$$

Then

$$\mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \mathcal{F}_3 \, .$$

*Proof.* We have $P, Q \in \mathcal{P}$ such that $\mathcal{F}_1 \vee \mathcal{F}_2$ and $\mathcal{F}_3$ are $P$-independent, while $\mathcal{F}_1$ and $\mathcal{F}_2 \vee \mathcal{F}_3$ are $Q$-independent. Consider the $\mathcal{F}_1 \vee \mathcal{F}_2$-measurable density[8]

$$f_{12} = \frac{Q|_{\mathcal{F}_1 \vee \mathcal{F}_2}}{P|_{\mathcal{F}_1 \vee \mathcal{F}_2}}$$

and the measure $R = f_{12} \cdot P \in \mathcal{P}$, then $R|_{\mathcal{F}_1 \vee \mathcal{F}_2} = Q|_{\mathcal{F}_1 \vee \mathcal{F}_2}$. The $P$-independence of $\mathcal{F}_1 \vee \mathcal{F}_2$ and $\mathcal{F}_3$ implies their $R$-independence. For every $A \in \mathcal{F}_1$, $B \in \mathcal{F}_2$, $C \in \mathcal{F}_3$

$$\begin{aligned} R(A \cap B \cap C) = R(A \cap B) R(C) = Q(A \cap B) R(C) = \\ = Q(A) Q(B) R(C) = R(A) R(B) R(C) \, . \end{aligned}$$

□

[8]In fact, it is a conditional expectation w.r.t. $P$, $f_{12} = \mathbb{E} \left( \frac{Q}{P} \,\middle|\, \mathcal{F}_1 \vee \mathcal{F}_2 \right)$.

The proof of Proposition 3.1 is now complete. If you find the proof of Lemma 3.5 rather tricky, consider the following calculation as a clue. Let $\psi_k \in L_2(\Omega, \mathcal{F}_k, \mathcal{P})$ $(k = 1, 2, 3)$, then (writing for short $\mathcal{F}_{12} = \mathcal{F}_1 \vee \mathcal{F}_2$ etc.),

$$
\begin{aligned}
(\psi_1 \otimes \psi_2) \otimes \psi_3 = \frac{\psi_1 \otimes \psi_2}{\sqrt{P|_{\mathcal{F}_{12}}}} \cdot \frac{\psi_3}{\sqrt{P|_{\mathcal{F}_3}}} \cdot \sqrt{P} = \\
= \frac{\psi_1}{\sqrt{Q|_{\mathcal{F}_1}}} \cdot \frac{\psi_2}{\sqrt{Q|_{\mathcal{F}_2}}} \cdot \sqrt{\frac{Q|_{\mathcal{F}_{12}}}{P|_{\mathcal{F}_{12}}}} \cdot \frac{\psi_3}{\sqrt{P|_{\mathcal{F}_3}}} \cdot \sqrt{P} \, ,
\end{aligned}
$$

and the equality $\|(\psi_1 \otimes \psi_2) \otimes \psi_3\|^2 = \|\psi_1\|^2 \|\psi_2\|^2 \|\psi_3\|^2$ turns into the following equality for functions $f_1 = (\psi_1 / \sqrt{Q|_{\mathcal{F}_1}})^2$, $f_2 = (\psi_2 / \sqrt{Q|_{\mathcal{F}_2}})^2$, $f_3 = (\psi_3 / \sqrt{P|_{\mathcal{F}_3}})^2$:

$$
\int f_1 f_2 f_3 \frac{Q|_{\mathcal{F}_{12}}}{P|_{\mathcal{F}_{12}}} \, dP = \Big( \int f_1 \, dQ|_{\mathcal{F}_1} \Big) \Big( \int f_2 \, dQ|_{\mathcal{F}_2} \Big) \Big( \int f_3 \, dP|_{\mathcal{F}_3} \Big) .
$$

The argument is easily generalized for $\mathcal{F}_1, \dots, \mathcal{F}_n$. Now we turn to infinite sequences of subspaces.

**3.6. Definition.** Let $X$ be a metrizable topological space and $X_1, X_2, \dots \subset X$ closed subsets. We define $\liminf_{n\to\infty} X_n$ as the set of limits of all convergent sequences $x_1, x_2, \dots$ such that $x_1 \in X_1, x_2 \in X_2, \dots$

The set $\liminf X_n$ is always closed. If $X$ is a linear topological space and $X_n$ are linear subspaces, then $\liminf X_n$ is a linear subspace. If $X$ is the $\sigma$-field $\mathcal{F}$ of a measure type space $(\Omega, \mathcal{F}, \mathcal{P})$ and $X_n$ are sub-$\sigma$-fields, then $\liminf X_n$ is a sub-$\sigma$-field.[9] Proofs of these facts are left to the reader.

**3.7. Proposition.** Let $H$ be an FHS-space, $E_n, F_n \subset H$ subspaces $(n = 1, 2, \dots)$, and $\liminf E_n = H$. For each $n$ denote by $\mathcal{N}_n$ the set of all admissible norms on $H$ that make $E_n, F_n$ orthogonal. Then the set $\liminf \mathcal{N}_n$ is either the empty set, or the whole $\mathcal{N}$.

*Proof.* We identify $H$ with $G_0$ of a Gaussian type space $(\Omega, \mathcal{F}, \mathcal{P}, G)$ and use the natural homeomorphism $\gamma \leftrightarrow \|\cdot\|_\gamma$ between the space $\mathcal{N}$ of admissible norms and the space $\mathcal{P}_G$ of Gaussian measures. Subspaces $E_n, F_n$ generate corresponding sub-$\sigma$-fields $\mathcal{E}_n, \mathcal{F}_n \subset \mathcal{F}$. The set $\mathcal{N}_n \subset \mathcal{N}$ corresponds to $\mathcal{P}_n \cap \mathcal{P}_G$, where $\mathcal{P}_n$ consists of all measures making $\mathcal{E}_n, \mathcal{F}_n$ independent. The following two lemmas complete the proof. ∎

[9] The natural topology of $\mathcal{F}$ is defined by a metric $\operatorname{dist}(A, B) = P(A \setminus B) + P(B \setminus A)$; the metric depends on $P \in \mathcal{P}$, but the topology does not. Of course, $X$ is $\mathcal{F}$ mod 0 rather than $\mathcal{F}$ itself.

**3.8. Lemma.** Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a measure type space, $E, E_1, E_2, \ldots \subset L_0(\mathcal{P})$ closed linear subspaces, and $\mathcal{E}, \mathcal{E}_1, \mathcal{E}_2, \ldots \subset \mathcal{F}$ the sub-$\sigma$-fields generated by the subspaces. Then

$$\liminf E_n = E \quad \text{implies} \quad \liminf \mathcal{E}_n \supset \mathcal{E} \,.$$

*Proof.* We have to prove that every $f \in \liminf E_n$ is measurable w.r.t. $\liminf \mathcal{E}_n$. It suffices to prove that the set $A = \{\omega : f(\omega) \le a\}$ belongs to $\liminf \mathcal{E}_n$ for every $a$ such that the set $\{\omega : f(\omega) = a\}$ is negligible (indeed, such $a$ are dense in $\mathbb{R}$). We take $f_n \in E_n$ such that $f_n \to f$ in $L_0(\mathcal{P})$, consider sets $A_n = \{\omega : f_n(\omega) \le a\}$ and note that $A_n \in \mathcal{E}_n$ and $A_n \to A$. $\square$

Note. In general, $\liminf \mathcal{E}_n$ need not be equal to $\mathcal{E}$; it may happen that $\mathcal{E}_n = \mathcal{F}$ but $\liminf E_n = \{0\}$ (even for one-dimensional $E_n$). However, in the Gaussian case (that is, when $(\Omega, \mathcal{F}, \mathcal{P}, G)$ is a Gaussian type space, and each $E_n$ is a Gaussian space) the equality $\liminf \mathcal{E}_n = \mathcal{E}$ holds, which is neither proved nor used here.

**3.9. Lemma.** Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a measure type space, $\mathcal{E}_n, \mathcal{F}_n \subset \mathcal{F}$ sub-$\sigma$-fields $(n = 1, 2, \ldots)$, and $\liminf \mathcal{E}_n = \mathcal{F}$. For each $n$ denote by $\mathcal{P}_n$ the set of all $P \in \mathcal{P}$ such that $\mathcal{E}_n$ and $\mathcal{F}_n$ are $P$-independent. Then

(a) the set $\liminf \mathcal{P}_n$ is either the empty set or the whole $\mathcal{P}$;

(b) if $\liminf \mathcal{P}_n = \mathcal{P}$ then $\|(P - Q)|_{\mathcal{F}_n}\| \to 0$ for all $P, Q \in \mathcal{P}$ (here $\| \cdot \|$ means the total variation).

*Proof.* Assume that $\liminf \mathcal{P}_n$ is nonempty; we have $P_n, P \in \mathcal{P}$ such that $P_n \to P$ and $\mathcal{E}_n, \mathcal{F}_n$ are $P_n$-independent. Let $K \subset (0, \infty)$ be a finite set and $f : \Omega \to K$ an $\mathcal{F}$-measurable function satisfying $\int f \, dP = 1$. Take $\mathcal{E}_n$-measurable functions $f_n : \Omega \to K$ such that $f_n \to f$ in $L_0(\mathcal{P})$ and $\int f_n \, dP = 1$. The $P_n$-independent $\sigma$-fields $\mathcal{E}_n, \mathcal{F}_n$ are also $Q_n$-independent, where $Q_n = f_n \cdot P_n \to f \cdot P = Q$ (since $\|f_n \cdot P_n - f_n \cdot P\| \le \|f_n\|_\infty \|P_n - P\| \to 0$). Thus $Q_n \in \mathcal{P}_n$ and $Q \in \liminf \mathcal{P}_n$. However, such measures $Q$ (for all $f$ and $K$) are dense in $\mathcal{P}$. So, $\liminf \mathcal{P}_n = \mathcal{P}$, which is (a). Also, the $P_n$-independence of $\mathcal{E}_n, \mathcal{F}_n$ implies $Q_n|_{\mathcal{F}_n} = P_n|_{\mathcal{F}_n}$. However, $\|(P_n - P)|_{\mathcal{F}_n}\| \le \|P_n - P\| \to 0$, and similarly $\|(Q_n - Q)|_{\mathcal{F}_n}\| \to 0$. So, $\|(P - Q)|_{\mathcal{F}_n}\| \to 0$ for all $Q$ of a dense set, therefore for all $Q$, which is (b). $\square$

The proof of Proposition 3.7 is now complete.

**3.10. Definition.** (a) Let $H$ be an FHS-space, $E_n, F_n \subset H$ subspaces, and $\liminf E_n = H$. We say that $F_n$ is asymptotically orthogonal to $E_n$, if there exists a convergent[10] sequence of admissible norms $\| \cdot \|_n$ such that for each $n$, $F_n$ is orthogonal to $E_n$ w.r.t. $\| \cdot \|_n$.

[10]In the space $\mathcal{N}$ of all admissible norms, whose topology is defined in Sect. 2.

(b) Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a measure type space, $\mathcal{E}_n, \mathcal{F}_n \subset \mathcal{F}$ sub-$\sigma$-fields, and $\liminf \mathcal{E}_n = \mathcal{F}$. We say that $\mathcal{F}_n$ is asymptotically independent of $\mathcal{E}_n$, if there exists a convergent[11] sequence of measures $P_n \in \mathcal{P}$ such that for each $n$, $\mathcal{F}_n$ is independent of $\mathcal{E}_n$ w.r.t. $P_n$.

Proposition 3.7 states that the norms $\|\cdot\|_n$ can be chosen so as to converge to any given admissible norm, provided that $F_n$ is asymptotically orthogonal to $E_n$. Similarly, if $\mathcal{F}_n$ is asymptotically independent of $\mathcal{E}_n$, then the measures $P_n$ can be chosen so as to converge to any given $P \in \mathcal{P}$ due to 3.9(a), and $\|(P-Q)|_{\mathcal{F}_n}\| \to 0$ for all $P, Q \in \mathcal{P}$ due to 3.9(b). Also, in the case of a Gaussian type space $(\Omega, \mathcal{F}, \mathcal{P}, G)$ and $H = G_0$, asymptotical orthogonality of subspaces is equivalent to asymptotical independence of the corresponding sub-$\sigma$-fields.

**3.11. Lemma.** Let $H$ be an FHS-space, $F_n \subset H$ subspaces, then the following conditions are equivalent.

(a) For every $f_1 \in F_1$, $f_2 \in F_2$, $\dots$ , if the sequence $(f_n)$ is bounded then $f_n \to 0$ weakly.[12]

(b) For every finite-dimensional subspaces $E_1, E_2, \dots$ such that $\liminf E_n = H$ there exist integers $k_1 \le k_2 \le \dots$ such that $k_n \to \infty$ and $F_n$ is asymptotically orthogonal to $E_{k_n}$.

*Proof.* We choose an admissible norm on $H$, thus turning $H$ into a Hilbert space. Condition (a) becomes

$$\forall h \in H \qquad \sup_{f \in F_n, \|f\| \le 1} \langle f, h \rangle \xrightarrow[n\to\infty]{} 0 \,,$$

that is,

$$\forall h \in H \quad \angle(h, F_n) \xrightarrow[n\to\infty]{} \frac{\pi}{2} \,,$$

where the angle is defined by $\cos \angle(h, F_n) = \sup\{\langle h, f\rangle : f \in F_n, \|f\| \le 1\}$. It is equivalent to

$$\forall E \quad \angle(E, F_n) \xrightarrow[n\to\infty]{} \frac{\pi}{2} \,,$$

where $E$ runs over finite-dimensional subspaces, and $\cos \angle(E, F_n) = \sup\{\langle e, f\rangle : e \in E, f \in F_n, \|e\| \le 1, \|f\| \le 1\}$. The following lemma completes the proof, provided that $k_n$ tends to $\infty$ slowly enough. Namely, in terms of $\delta(\cdot,\cdot)$ introduced there, it suffices that $\delta\big(\angle(E_{k_n}, F_n), \dim E_{k_n}\big) \xrightarrow[n\to\infty]{} 0$. $\square$

[11] In the space $\mathcal{P}$ whose topology is defined in Sect. 1.

[12] That is, $\langle f_n, h\rangle \to 0$ for every $h \in H$.

**3.12. Lemma.** Let $H$ be an FHS-space, $E, F \subset H$ subspaces, $\dim(E) < \infty$, $E \cap F = \{0\}$. Then for every admissible norm $\|\cdot\|_1$ there exists an admissible norm $\|\cdot\|_2$ such that $E, F$ are orthogonal w.r.t. $\|\cdot\|_2$, and

$$\operatorname{dist}(\|\cdot\|_1, \|\cdot\|_2) \le \delta\bigl(\angle(E,F), \dim E\bigr)$$

for some function $\delta : [0, \frac{\pi}{2}] \times \{0, 1, 2, \dots\} \to (0, \infty)$ such that for every $n$, $\delta(\alpha, n) \to 0$ for $\alpha \to \frac{\pi}{2}$.[13]

*Proof.* We equip $H$ with the norm $\|\cdot\|_1$, thus turning $H$ into a Hilbert space, and consider orthogonal projections $Q_E, Q_F$ onto $E, F$ respectively. Introduce subspaces $E \cap F^\perp$, $E^\perp \cap F$, $E^\perp \cap F^\perp$ (here $E^\perp$ is the orthogonal complement of $E$); the subspaces are orthogonal to each other, and invariant under both $Q_E$ and $Q_F$. Therefore

$$H = H_0 \oplus (E \cap F^\perp) \oplus (E^\perp \cap F) \oplus (E^\perp \cap F^\perp)\,,$$

where $H_0$ is another subspace invariant under $Q_E, Q_F$ (since these operators are Hermitian). Introduce $E_0 = E \cap H_0$, $F_0 = F \cap H_0$, then $Q_E h_0 = Q_{E_0} h_0$ for all $h_0 \in H_0$ (since $Q_E$ commutes with $Q_{H_0}$), and $Q_F h_0 = Q_{F_0} h_0$. We may get rid of $H_0^\perp$ by letting

$$\|h_0 + h_1\|_2^2 = \|h_0\|_2^2 + \|h_1\|_1^2 \quad \text{for all } h_0 \in H_0,\ h_1 \in H_0^\perp.$$

In other words, we'll construct $\|\cdot\|_2$ on $H_0$ while preserving both the given norm on $H_1$ and the orthogonality of $H_0, H_1$. Now we forget about $H_1$, assuming that $H = H_0$, $E = E_0$, $F = F_0$.

So, we have $E \cap F = \{0\}$, $E \cap F^\perp = \{0\}$, $E^\perp \cap F = \{0\}$, $E^\perp \cap F^\perp = \{0\}$. The latter implies $\dim(F^\perp) \le \operatorname{codim}(E^\perp) = \dim E$. Similarly, $\dim F \le \dim E$. Therefore $H$ is finite-dimensional, $\dim H \le 2 \dim E$.

Both $Q_E$ and $Q_F$ commute with the Hermitian operator $C = \frac{1}{2}(2Q_E - 1)(2Q_F - 1) + \frac{1}{2}(2Q_F - 1)(2Q_E - 1)$. The spectrum of $C$ consists of some numbers $\cos 2\varphi_k$ of multiplicity 2 (though, some $\varphi_k$ may coincide), and $0 < \varphi_k < \frac{\pi}{2}$ (the case $\varphi_k = 0$ is excluded by $E \cap F = \{0\}$; the case $\varphi_k = \pi/2$ is excluded by $E \cap F^\perp = \{0\}$, $E^\perp \cap F = \{0\}$, $E^\perp \cap F^\perp = \{0\}$). Accordingly, $H_0$ decomposes into the (orthogonal) direct sum of planes, $H = H_1 \oplus \dots \oplus H_d$, $\dim H_k = 2$, invariant under $Q_E, Q_F$. Subspaces $E_k = E \cap H_k$, $F_k = F \cap H_k$ are two lines on the plane $H_k$, and $\angle(E_k, F_k) = \varphi_k$; $k = 1, \dots, d$; $d \le \dim E$. Clearly,

$$\angle(E, F) = \min(\varphi_1, \dots, \varphi_d)\,.$$

[13]Of course, $\delta$ does not depend on $H, E, F$.

We construct $\|\cdot\|_2$ on each $H_k$ separately, while preserving their orthogonality. Elementary 2-dimensional geometry shows that the corresponding numbers $\lambda'_k, \lambda''_k$ (two numbers for each plane) are, in the optimal case,

$$\lambda'_k = \bigl(\tan\frac{\varphi}{2}\bigr)^{-1/2}, \quad \lambda''_k = \bigl(\tan\frac{\varphi}{2}\bigr)^{1/2}.$$

The corresponding angle $\beta$ between $\sqrt{\gamma_1}$ and $\sqrt{\gamma_2}$ is given by

$$\cos\beta = \prod_{k=1}^{d} \biggl( \frac{\tan^{-1/4}\frac{\varphi}{2} + \tan^{1/4}\frac{\varphi}{2}}{2} \biggr)^{-1},$$

therefore

$$\beta \le \arccos \biggl( \frac{\tan^{-1/4}\frac{\alpha}{2} + \tan^{1/4}\frac{\alpha}{2}}{2} \biggr)^{-\dim E};$$

here $\beta = \angle(\sqrt{\gamma_1}, \sqrt{\gamma_2}) = \operatorname{dist}(\|\cdot\|_1, \|\cdot\|_2)$, $\alpha = \angle(E, F)$. $\square$

**3.13. Definition.** Let $H$ be an FHS-space, $F_n \subset H$ subspaces. We write $\limsup F_n = \{0\}$, if the sequence $(F_n)$ satisfies equivalent conditions (a), (b) of Lemma 3.11.

Note. More generally, one could define $\limsup F_n$ as the set of limits of all weakly convergent subsequences of all bounded sequences $f_1, f_2, \dots$ such that $f_1 \in F_1, f_2 \in F_2, \dots$ It is in general not a linear space, but anyway, $(\limsup E_n^\perp)^\perp = \liminf E_n$ in a Hilbert space. For an FHS-space, as well as a separable Banach space, $F_n$ should be situated in the dual space. However, all that is not needed here.

**3.14. Theorem.** Let $H$ be an FHS-space, $E_n, F_n \subset H$ subspaces ($n = 1, 2, \dots$) such that $\liminf E_n = H$, and $\limsup F_n = \{0\}$, and $H = E_n \oplus F_n$ (in the FHS sense) for all $n$. Then there exist subspaces $G_n, H_n \subset H$ such that

$$E_n = G_n \oplus H_n \quad \text{and} \quad H = G_n \oplus H_n \oplus F_n$$

(both in the FHS sense), and $\liminf G_n = H$, and $H_n \oplus F_n$ is asymptotically orthogonal to $G_n$.

*Proof.* We choose an admissible norm on $H$, thus turning $H$ into a Hilbert space. Let $L \subset H$ be a finite-dimensional subspace, $L \ne \{0\}$. For any given $n$ consider the pair $L, E_n$. Its geometry may be described (similarly to the

proof of Lemma 3.12) via angles $\varphi_1^{(n)}, \dots, \varphi_{d_n}^{(n)} \in [0, \frac{\pi}{2})$, $d_n \le \dim L$. This time, zero angles are allowed, since $L \cap E_n$ need not be $\{0\}$. It may happen that $d_n < \dim L$, since $L \cap E_n^\perp$ need not be $\{0\}$. However,

$$\sup_{x \in L, x \ne 0} \angle(x, E_n) = \alpha_n \to 0 \quad \text{for } n \to \infty\,;$$

for large $n$ we have $\alpha_n < \pi/2$ which implies $d_n = d = \dim L$ and $\max(\varphi_1^{(n)}, \dots, \varphi_d^{(n)}) = \alpha_n$. We may send $L$ into $E_n$ rotating it by $\varphi_1^{(n)}, \dots, \varphi_d^{(n)}$. In other words, there is a rotation $U_n : H \to H$ such that

$$U_n(L) \subset E_n \quad \text{and} \quad \|U_n - 1\| \le 2 \sin \frac{\alpha_n}{2} \xrightarrow[n\to\infty]{} 0\,.$$

We choose subspaces $L_k \subset H$ such that $\dim L_k = k$ and $\liminf L_k = H$.[14] Introduce

$$\alpha_{k,n} = \sup_{x \in L_k, x \ne 0} \angle(x, E_n)\,,$$

then $\alpha_{k,n} \xrightarrow[n\to\infty]{} 0$ for each $k$. On the other hand, introduce

$$\beta_{k,n} = \frac{\pi}{2} - \angle(L_k, F_n)\,.$$

Similarly to the proof of Lemma 3.11 we have $\beta_{k,n} \xrightarrow[n\to\infty]{} 0$ for each $k$, therefore[15] $\delta(\frac{\pi}{2} - \beta_{k_n,n}, k_n) \xrightarrow[n\to\infty]{} 0$ if $k_n$ tends to $\infty$ slowly enough. However, we choose $k_1 \le k_2 \le \cdots$, $k_n \to \infty$ so as to satisfy a stronger condition:

$$\delta\Big(\frac{\pi}{2} - \alpha_{k_n,n} - \beta_{k_n,n},\, k_n\Big) \xrightarrow[n\to\infty]{} 0\,.$$

We take

$$G_n = U_n(L_{k_n})\,,$$

where rotations $U_n$ satisfy $U_n(L_{k_n}) \subset E_n$ and $\|U_n - 1\| \le 2\sin(\frac{1}{2}\alpha_{k_n,n}) \to 0$. Then $\liminf G_n = H$, and

$$\frac{\pi}{2} - \angle(G_n, F_n) \le \alpha_{k_n,n} + \beta_{k_n,n}\,;$$

due to Lemma 3.12, $F_n$ is asymptotically orthogonal to $G_n$. We take admissible norms $\|\cdot\|_n \to \|\cdot\|$ such that $F_n$ is orthogonal to $G_n$ w.r.t. $\|\cdot\|_n$.

[14] Of course, one may take $L_1 \subset L_2 \subset \cdots$

[15] Recall that $\delta(\cdot,\cdot)$ is introduced in Lemma 3.12.

Consider the orthogonal complement $M_n$ of $G_n$ w.r.t. $\|\cdot\|_n$; clearly, $M_n$ is asymptotically orthogonal to $G_n$. We have $F_n \subset M_n$ and $H = G_n \oplus M_n$ (in the FHS-sense). On the other hand, $G_n \subset E_n$ and $H = E_n \oplus F_n$. Proposition 3.1 states that the subspace

$$H_n = E_n \cap M_n$$

satisfies $G_n \oplus H_n \oplus F_n = H$ and $G_n \oplus H_n = E_n$ (and also $H_n \oplus F_n = M_n$). $\square$

# 4 Density matrices

Recall the notion of a density matrix (borrowed from quantum theory). Let $H_1, H_2$ be Hilbert spaces, $H = H_1 \otimes H_2$, and $\psi \in H$, $\|\psi\| = 1$. Every unit vector $\xi \in H_1$ determines a subspace $\xi \otimes H_2 \subset H$ and the corresponding projection operator $Q_{\xi\otimes H_2} = Q_\xi \otimes \mathbf{1}_{H_2}$; here $Q_\xi : H_1 \to H_1$, $Q_\xi x = (x, \xi)\xi$ is a one-dimensional projection, and $\mathbf{1}_{H_2} : H_2 \to H_2$, $\mathbf{1}_{H_2} y = y$ the unit operator. The function

$$\xi \mapsto \|Q_{\xi\otimes H_2}\psi\|^2$$

is a quadratic form on $H_1$. The corresponding operator $\rho_\psi : H_1 \to H_1$ satisfies

$$\langle \rho_\psi \xi, \xi \rangle = \|Q_{\xi\otimes H_2}\psi\|^2 = \langle (Q_\xi \otimes \mathbf{1}_{H_2})\psi, \psi \rangle$$

(that is, $\langle \rho_\psi \rangle_\xi = \langle Q_\xi \otimes \mathbf{1}_{H_2} \rangle_\psi$) for all $\xi \in H_1$; one calls $\rho_\psi$ the *density matrix* of $\psi$ (on $H_1$). In terms of an orthonormal basis $(e_k)$ of $H_2$,

$$\psi = \sum_k \psi_k \otimes e_k \quad \text{for some } \psi_k \in H_1 \,;$$

$$(Q_\xi \otimes \mathbf{1}_{H_2})\psi = \sum_k (Q_\xi \psi_k) \otimes e_k = \sum_k \langle \psi_k, \xi \rangle \xi \otimes e_k \,;$$

$$\langle \rho_\psi \xi, \xi \rangle = \sum_k |\langle \psi_k, \xi \rangle|^2 \,.$$

Note that

$$\rho_\psi \ge 0 \,; \qquad \operatorname{Tr}(\rho_\psi) = 1 \,;$$

$$\operatorname{Tr}\big((\rho_{\psi_1} - \rho_{\psi_2})A\big) \le 2\|\psi_1 - \psi_2\| \cdot \|A\|$$

for all unit vectors $\psi_1, \psi_2 \in H$ and operators $A : H_1 \to H_1$. The inequality may be proven as follows: $\operatorname{Tr}(\rho_\psi Q_\xi) = \operatorname{Tr}\big((Q_\xi \otimes \mathbf{1}_{H_2})Q_\psi\big)$; $\operatorname{Tr}(\rho_\psi A) = \operatorname{Tr}\big((A \otimes$

$\mathbf{1}_{H_2})Q_\psi\big)$; $\operatorname{Tr}\big((\rho_{\psi_1}-\rho_{\psi_2})A\big) = \operatorname{Tr}\big((A\otimes\mathbf{1}_{H_2})(Q_{\psi_1}-Q_{\psi_2})\big) \le \|A\otimes\mathbf{1}_{H_2}\|\cdot\|Q_{\psi_1}-Q_{\psi_2}\| \le \|A\|\cdot 2\|\psi_1-\psi_2\|$.

Note also that

$$\psi' = (U_1\otimes U_2)\psi \quad \text{implies} \quad \rho_{\psi'} = U_1\rho_\psi U_1^*$$

for all unitary operators $U_1 : H_1 \to H_1$, $U_2 : H_2 \to H_2$. Proof: $\langle\rho_{\psi'}\xi,\xi\rangle = \langle(Q_\xi\otimes\mathbf{1}_{H_2})\psi',\psi'\rangle = \langle(U_1\otimes U_2)^*(Q_\xi\otimes\mathbf{1}_{H_2})(U_1\otimes U_2)\psi,\psi\rangle = \langle(U_1^*Q_\xi U_1\otimes U_2^*\mathbf{1}_{H_2}U_2)\psi,\psi\rangle = \langle(Q_{U_1^*\xi}\otimes\mathbf{1}_{H_2})\psi,\psi\rangle = \langle\rho_\psi U_1^*\xi, U_1^*\xi\rangle = \langle U_1\rho_\psi U_1^*\xi,\xi\rangle$ for all unit vectors $\xi\in H_1$, $\psi\in H$.

Assume in addition that $H_1 = L_2(\Omega_1,\mathcal{F}_1,P_1)$, $H_2 = L_2(\Omega_2,\mathcal{F}_2,P_2)$, then (after the usual identification) $H = L_2(\Omega,\mathcal{F},P)$ where $(\Omega,\mathcal{F},P) = (\Omega_1,\mathcal{F}_1,P_1)\otimes(\Omega_2,\mathcal{F}_2,P_2)$. We have

$$\langle\rho_\psi\xi,\xi\rangle = \iint \xi(\omega')\overline{\xi(\omega'')}\rho_\psi(\omega',\omega'')\,P_1(d\omega')P_1(d\omega'')\,,$$

where $\rho_\psi$ is an element of $L_2\big((\Omega_1,\mathcal{F}_1,P_1)\otimes(\Omega_1,\mathcal{F}_1,P_1)\big)$ (the kernel of the operator $\rho_\psi : H_1\to H_1$), namely,

$$\rho_\psi(\omega',\omega'') = \int \overline{\psi(\omega',\omega_2)}\psi(\omega'',\omega_2)\,P_2(d\omega_2)\,.$$

In terms of a basis,

$$\psi(\omega_1,\omega_2) = \sum_k \psi_k(\omega_1)e_k(\omega_2)\,;$$
$$\rho_\psi(\omega',\omega'') = \sum_k \overline{\psi_k(\omega')}\psi_k(\omega'')\,.$$

The proof is basically a calculation:

$$\begin{aligned}\rho_\psi(\omega',\omega'') &= \int \overline{\psi(\omega',\omega_2)}\psi(\omega'',\omega_2)\,P_2(d\omega_2) = \\ &= \int \overline{\sum_k \psi_k(\omega')e_k(\omega_2)}\sum_l \psi_l(\omega'')e_l(\omega_2)\,P_2(d\omega_2) = \\ &= \sum_{k,l}\overline{\psi_k(\omega')}\psi_l(\omega'')\int\overline{e_k(\omega_2)}e_l(\omega_2)\,P_2(d\omega_2) = \sum_k \overline{\psi_k(\omega')}\psi_k(\omega'')\,;\end{aligned}$$

$$\begin{aligned}\langle\rho_\psi\xi,\xi\rangle = \sum_k |\langle\psi_k,\xi\rangle|^2 = \sum_k \Big|\int \psi_k(\omega)\overline{\xi(\omega)}\,P_1(d\omega)\Big|^2 = \\ = \sum_k \Big(\int \overline{\psi_k(\omega')}\xi(\omega')\,P_1(d\omega')\cdot\int \psi_k(\omega'')\overline{\xi(\omega'')}\,P_1(d\omega'')\Big) = \\ = \iint \xi(\omega')\overline{\xi(\omega'')}\Big(\sum_k \overline{\psi_k(\omega')}\psi_k(\omega'')\Big)\,P_1(d\omega')P_1(d\omega'') = \\ = \iint \xi(\omega')\overline{\xi(\omega'')}\rho_\psi(\omega',\omega'')\,P_1(d\omega')P_1(d\omega'')\,.\end{aligned}$$

We turn to measure type spaces $(\Omega,\mathcal{F},\mathcal{P}) = (\Omega_1,\mathcal{F}_1,\mathcal{P}_1)\otimes(\Omega_2,\mathcal{F}_2,\mathcal{P}_2)$ (which means $P_1\otimes P_2\in\mathcal{P}$ for some, therefore all, $P_1\in\mathcal{P}_1$ and $P_2\in\mathcal{P}_2$), and the corresponding Hilbert spaces; $H = L_2(\Omega,\mathcal{F},\mathcal{P}) = L_2(\Omega_1,\mathcal{F}_1,\mathcal{P}_1)\otimes L_2(\Omega_2,\mathcal{F}_2,\mathcal{P}_2) = H_1\otimes H_2$ under the natural identification

$$\frac{\psi_1\otimes\psi_2}{\sqrt{P_1\otimes P_2}}(\omega_1,\omega_2) = \frac{\psi_1}{\sqrt{P_1}}(\omega_1)\cdot\frac{\psi_2}{\sqrt{P_2}}(\omega_2)\,,$$

that is,

$$\psi_1\otimes\psi_2 = \Big(\frac{\psi_1}{\sqrt{P_1}}\otimes\frac{\psi_2}{\sqrt{P_2}}\Big)\cdot\sqrt{P_1\otimes P_2}\in L_2(\mathcal{P})$$

for all $\psi_1\in L_2(\mathcal{P}_1)$, $\psi_2\in L_2(\mathcal{P}_2)$, $P_1\in\mathcal{P}_1$, $P_2\in\mathcal{P}_2$. An element $\psi\in H$, $\|\psi\|=1$, determines an operator $\rho_\psi : H_1\to H_1$, whose kernel (denoted also by $\rho_\psi$) belongs to $L_2\big((\Omega_1,\mathcal{F}_1,\mathcal{P}_1)\otimes(\Omega_1,\mathcal{F}_1,\mathcal{P}_1)\big)$;

$$\langle\rho_\psi\xi,\xi\rangle = \iint \frac{\xi}{\sqrt{P_1}}(\omega')\frac{\xi}{\sqrt{P_1}}(\omega'')\frac{\rho_\psi}{\sqrt{P_1\otimes P_1}}(\omega',\omega'')\,P_1(d\omega')P_1(d\omega'')\,,$$

$$\frac{\rho_\psi}{\sqrt{P_1\otimes P_1}}(\omega',\omega'') = \int \overline{\frac{\psi}{\sqrt{P_1\otimes P_2}}(\omega',\omega_2)}\cdot\frac{\psi}{\sqrt{P_1\otimes P_2}}(\omega'',\omega_2)\,P_2(d\omega_2)$$

for all $\xi\in L_2(\mathcal{P}_1)$, $\|\xi\|=1$, and $P_1\in\mathcal{P}_1$, $P_2\in\mathcal{P}_2$.

Let $(\Omega,\mathcal{F},\mathcal{P})$ be a measure type space and $\mathcal{F}_1,\mathcal{F}_2\subset\mathcal{F}$ sub-$\sigma$-fields such that $\mathcal{F}=\mathcal{F}_1\otimes\mathcal{F}_2$ (as defined by 3.3), then

$$L_2(\mathcal{F}) = L_2(\mathcal{F}_1)\otimes L_2(\mathcal{F}_2)\,.$$

Here $L_2(\mathcal{F}) = L_2(\Omega,\mathcal{F},\mathcal{P})$ and $L_2(\mathcal{F}_1) = L_2(\Omega,\mathcal{F}_1,\mathcal{P}|_{\mathcal{F}_1})$, the same for $\mathcal{F}_2$,[16] and the natural identification is made, namely,

$$\big(f_1\cdot\sqrt{P|_{\mathcal{F}_1}}\big)\otimes\big(f_2\cdot\sqrt{P|_{\mathcal{F}_2}}\big) = (f_1f_2)\cdot\sqrt{P}$$

[16]Of course, $\mathcal{P}|_{\mathcal{F}_1} = \{P|_{\mathcal{F}_1} : P\in\mathcal{P}\}$ consists of restricted measures. Alternatively one may introduce quotient spaces $\Omega_1=\Omega/\mathcal{F}_1$, $\Omega_2=\Omega/\mathcal{F}_2$ and identify $\Omega$ with $\Omega_1\times\Omega_2$.

whenever $f_1 \in L_2(\Omega, \mathcal{F}_1, P)$, $f_2 \in L_2(\Omega, \mathcal{F}_2, P)$, and $P \in \mathcal{P}$ makes $\mathcal{F}_1, \mathcal{F}_2$ independent. We need a counterpart of Lemma 3.9.

**4.1. Lemma.** Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a measure type space, $\mathcal{E}_n, \mathcal{F}_n \subset \mathcal{F}$ sub-$\sigma$-fields $(n = 1, 2, \dots)$, $\mathcal{F} = \mathcal{E}_n \otimes \mathcal{F}_n$ for each $n$, and $\liminf \mathcal{E}_n = \mathcal{F}$, and $\mathcal{F}_n$ is asymptotically independent of $\mathcal{E}_n$. Then for every $P \in \mathcal{P}$

$$\liminf \bigl( L_2(\mathcal{E}_n) \otimes \sqrt{P|_{\mathcal{F}_n}} \bigr) = L_2(\mathcal{F}) \, .$$

That is, for every $\psi \in L_2(\mathcal{F})$ there exist $\xi_n \in L_2(\mathcal{E}_n)$ such that $\| \psi - \xi_n \otimes \sqrt{P|_{\mathcal{F}_n}} \| \to 0$ when $n \to \infty$.

*Proof.* Similarly to the proof of 3.9, we take $P_n \in \mathcal{P}$ such that $\mathcal{E}_n, \mathcal{F}_n$ are $P_n$-independent and $P_n \to P$. We consider an arbitrary finite set $K \subset \mathbb{R}$, an arbitrary $\mathcal{F}$-measurable function $f : \Omega \to K$, and the corresponding vector $\psi = f \sqrt{P} \in L_2(\mathcal{F})$. We construct $\mathcal{E}_n$-measurable functions $f_n : \Omega \to K$ such that $f_n \to f$ in $L_0(\mathcal{P})$, and corresponding vectors $\psi_n = f_n \cdot \sqrt{P_n} \in L_2(\mathcal{F})$. Independence of $\mathcal{E}_n, \mathcal{F}_n$ w.r.t. $P_n$ means that $\sqrt{P_n} = \sqrt{P_n|_{\mathcal{E}_n}} \otimes \sqrt{P_n|_{\mathcal{F}_n}}$, therefore $\psi_n = \xi_n \otimes \sqrt{P_n|_{\mathcal{F}_n}} \in L_2(\mathcal{E}_n) \otimes \sqrt{P_n|_{\mathcal{F}_n}}$, where $\xi_n = f_n \cdot \sqrt{P_n|_{\mathcal{E}_n}} \in L_2(\mathcal{E}_n)$. However, $\psi_n \to \psi$, since $\| f_n \sqrt{P_n} - f_n \sqrt{P} \| \le \| f_n \|_\infty \| \sqrt{P_n} - \sqrt{P} \| \to 0$. Also $\| P_n|_{\mathcal{F}_n} - P|_{\mathcal{F}_n} \| \le \| P_n - P \| \to 0$, therefore $\| \sqrt{P_n|_{\mathcal{F}_n}} - \sqrt{P|_{\mathcal{F}_n}} \| \to 0$. So, $\| \psi - \xi_n \otimes \sqrt{P|_{\mathcal{F}_n}} \| \le \| \psi - \psi_n \| + \| \xi_n \otimes \sqrt{P_n|_{\mathcal{F}_n}} - \xi_n \otimes \sqrt{P|_{\mathcal{F}_n}} \| \to 0$, and $\psi \in \liminf \bigl( L_2(\mathcal{E}_n) \otimes \sqrt{P|_{\mathcal{F}_n}} \bigr)$. It remains to note that such vectors $\psi$ (for all $f$ and $K$) are dense in $L_2(\mathcal{F})$. ☐

# 5 Fock spaces and tail density matrices

Recall the correspondence (described in Sect. 2) between Gaussian type spaces $(\Omega, \mathcal{F}, \mathcal{P}, G)$ and FHS-spaces $G_0 = G / \operatorname{Const}$. We know that any FHS-space $G_0$ determines (up to isomorphism) the corresponding Gaussian type space $(\Omega, \mathcal{F}, \mathcal{P}, G)$, which in turn determines the Hilbert space $H = L_2(\Omega, \mathcal{F}, \mathcal{P})$. I denote the relation by

$$H = \operatorname{Exp}(G_0) \, ,$$

and call $H$ the Fock exponential of $G_0$.[17] Why call it 'exponential'? Since

$$G_0 = G_1 \oplus G_2 \quad \text{implies} \quad \operatorname{Exp}(G_0) = \operatorname{Exp}(G_1) \otimes \operatorname{Exp}(G_2)$$

[17] By choosing an admissible norm on $G_0$ one turns $G_0$ into a Hilbert space, in which case $\operatorname{Exp}(G_0)$ contains a special element, 'ground state vector' $\sqrt{\gamma}$ (where $\gamma$ is the Gaussian measure corresponding to the chosen norm); that is the classical Fock construction.

in the following sense. Assume that $G_1, G_2 \subset G_0$ are subspaces such that $G_0 = G_1 \oplus G_2$ (in the FHS sense, as defined in Sect. 3). Then the sub-$\sigma$-fields $\mathcal{F}_1, \mathcal{F}_2 \subset \mathcal{F}$, generated by $G_1, G_2$ respectively, satisfy $\mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2$, therefore $L_2(\mathcal{F}) = L_2(\mathcal{F}_1) \otimes L_2(\mathcal{F}_2)$, as explained in Sect. 4. So, every decomposition of $G_0$ into a direct sum determines a decomposition of $\operatorname{Exp}(G_0)$ into a tensor product.

Given such a decomposition $G_0 = G_1 \oplus G_2$, every unit vector $\psi \in \operatorname{Exp}(G_0)$ determines a density matrix $\rho_\psi$ on $\operatorname{Exp}(G_2)$. Do not think, however, that $\rho_\psi$ is uniquely determined by $\psi$ and $G_2$; also $G_1$ influences $\rho_\psi$.

Consider an infinite sequence of decompositions, $G_0 = E_n \oplus F_n$, of a single FHS-space $G_0$. We want to know, whether or not $\|\rho_n(\psi_1) - \rho_n(\psi_2)\| \to 0$ when $n \to \infty$ for all unit vectors $\psi_1, \psi_2 \in \operatorname{Exp}(G_0)$; here $\rho_n(\psi)$ is the density matrix on $\operatorname{Exp}(F_n)$ that corresponds to $\psi \in \operatorname{Exp}(E_n) \otimes \operatorname{Exp}(F_n)$, and the trace norm is used,

$$\|\rho_n(\psi_1) - \rho_n(\psi_2)\| = \sup_{\|A\| \le 1} \operatorname{Tr}\bigl(\rho_n(\psi_1) - \rho_n(\psi_2))A\bigr) ;$$

here $A$ runs over Hermitian operators on $\operatorname{Exp}(F_n)$. In general we have only $\|\rho_n(\psi_1) - \rho_n(\psi_2)\| \le 2\|\psi_1 - \psi_2\|$.

Assume that $\liminf E_n = G_0$. If $F_n$ is asymptotically orthogonal to $E_n$, then $\|\rho_n(\psi_1) - \rho_n(\psi_2)\| \to 0$, which follows easily from Lemma 4.1. Namely, the lemma represents $\psi$ as the limit of $\xi_n \otimes \sqrt{P|_{\mathcal{F}_n}}$; however, $\rho_n(\xi_n \otimes \sqrt{P|_{\mathcal{F}_n}})$ is the one-dimensional projection $Q_n$ onto $\sqrt{P|_{\mathcal{F}_n}}$, therefore $\|\rho_n(\psi) - Q_n\| \le 2\|\psi - \xi_n \otimes \sqrt{P|_{\mathcal{F}_n}}\| \to 0$, and so, $\|\rho_n(\psi_1) - \rho_n(\psi_2)\| \le \|\rho_n(\psi_1) - Q_n\| + \|\rho_n(\psi_2) - Q_n\| \to 0$. However, the asymptotical orthogonality condition may be dropped, as we'll see now.

**5.1. Proposition.** Let $G_0$ be an FHS-space, $E_n, F_n \subset G_0$ subspaces such that $\liminf E_n = G_0$, and $\limsup F_n = \{0\}$, and $G_0 = E_n \oplus F_n$ (in the FHS sense) for all $n$. Then $\|\rho_n(\psi_1) - \rho_n(\psi_2)\| \to 0$ when $n \to \infty$, for all unit vectors $\psi_1, \psi_2 \in \operatorname{Exp}(G_0)$.

*Proof.* Theorem 3.14 gives us $G_n, H_n \subset G_0$ such that $E_n = G_n \oplus H_n$ and $G_0 = G_n \oplus H_n \oplus F_n$ (both in the FHS sense), and $\liminf G_n = G_0$, and $H_n \oplus F_n$ is asymptotically orthogonal to $G_n$. Lemma 4.1 gives us a representation

$$\psi = \lim_{n\to\infty} (\xi_n \otimes \chi_n)$$

for an arbitrary unit vector $\psi \in \operatorname{Exp}(G_0)$; here $\xi_n$ are unit vectors of $\operatorname{Exp}(G_n)$, $\chi_n$ are unit vectors of $\operatorname{Exp}(H_n \oplus F_n)$, and these $\chi_n$ (unlike $\xi_n$) do not depend on $\psi$. We have $\|\rho_n(\psi) - \rho_n(\xi_n \otimes \chi_n)\| \le 2\|\psi - \xi_n \otimes \chi_n\| \to 0$. However,

the density matrix $\rho_n(\xi_n \otimes \chi_n)$ on $\operatorname{Exp}(F_n)$ that corresponds to the vector $\xi_n \otimes \chi_n \in \operatorname{Exp}(G_n) \otimes \operatorname{Exp}(H_n) \otimes \operatorname{Exp}(F_n)$ is the same as the density matrix on $\operatorname{Exp}(F_n)$ that corresponds to the vector $\chi_n \in \operatorname{Exp}(H_n) \otimes \operatorname{Exp}(F_n)$. The vector does not depend on $\psi$, therefore $\rho_n(\xi_n \otimes \chi_n)$ does not depend on $\psi$. So, $\|\rho_n(\psi_1) - \rho_n(\psi_2)\| \le \|\rho_n(\psi_1) - \rho_n(\xi_n \otimes \chi_n)\| + \|\rho_n(\psi_2) - \rho_n(\xi_n \otimes \chi_n)\| \to 0$. $\square$

Consider a decomposition $G_0 = G_1 \oplus G_2$ of an FHS-space $G_0$ into the direct sum (in the FHS sense) of subspaces $G_1, G_2 \subset G_0$, and the corresponding decomposition $G^0 = G^1 \oplus G^2$ of its dual FHS-space $G^0$; that is, $G^1 = G_2^\perp$ is the annihilator of $G_2$ in $G^0$, and $G^2 = G_1^\perp$. Of course, also $G_1 = (G^2)^\perp$ and $G_2 = (G^1)^\perp$. Introduce an admissible norm $\|\cdot\|$ on $G_0$ (note that $G_1, G_2$ need not be orthogonal w.r.t. $\|\cdot\|$), and its dual norm on $G^0$ (denoted by $\|\cdot\|$ as well). Consider

$$\operatorname{dist}(g, G_2) = \inf_{g_2 \in G_2} \|g - g_2\| = \sup_{x_1 \in G^1, \|x_1\| \le 1} \langle g, x_1 \rangle \,,$$
$$\operatorname{dist}(x, G^2) = \inf_{x_2 \in G^2} \|x - x_2\| = \sup_{g_1 \in G_1, \|g_1\| \le 1} \langle g_1, x \rangle$$

for any $g \in G_0$, $x \in G^0$.

Introduce the corresponding Hilbert spaces $H = \operatorname{Exp}(G_0)$, $H_1 = \operatorname{Exp}(G_1)$, $H_2 = \operatorname{Exp}(G_2)$; we have $H = H_1 \otimes H_2$. Recall the operators $U_x$ and $V_g$ for $x \in G^0$, $g \in G_0$, satisfying the Canonical Commutation Relations $V_g U_x = e^{i\langle g,x \rangle} U_x V_g$. Let $(\Omega, \mathcal{F}, \mathcal{P}, G)$ be the corresponding Gaussian type space (thus, $H = L_2(\Omega, \mathcal{F}, \mathcal{P})$), and $\gamma \in \mathcal{P}_G$ a Gaussian measure such that $\|\cdot\|_\gamma = \|\cdot\|$. We know that the vector $\psi = \sqrt{\gamma} \in H$ satisfies $\langle U_x \psi, \psi \rangle = \exp\bigl(-\frac18 \|x\|^2\bigr)$, $\langle V_g \psi, \psi \rangle = \exp\bigl(-\frac12 \|g\|^2\bigr)$ for all $x \in G^0$, $g \in G_0$. Denote by $\rho(\psi)$ the corresponding density matrix on $H_1$. In the following lemma, $\rho(\psi')$ for some other $\psi'$ are the corresponding density matrices on $H_1$, and a function $M : [0,\infty) \to [0,\infty)$ is defined by[18]

$$M(r) = \max_{\varphi \in [0,\pi]} \biggl( \exp\Bigl( -\frac{\varphi^2}{2r^2} \Bigr) \cdot 2 \sin \frac{\varphi}{2} \biggr) \,.$$

**5.2. Lemma.** For all $x \in G^0$, $g \in G_0$

$$\|\rho(\psi) - \rho(U_x \psi)\| \ge M\bigl(\operatorname{dist}(x, G^2)\bigr) \,,$$
$$\|\rho(\psi) - \rho(V_g \psi)\| \ge M\bigl(2 \operatorname{dist}(g, G_2)\bigr) \,.$$

[18] The exact form of the function is of no importance here; we only need to know that $M(r_n) \to 0$ implies $r_n \to 0$.

*Proof.* Let $x = y + z$, $y \in G^1$, $z \in G^2$, then $U_x = U^{(1)}_y \otimes U^{(2)}_z$ (the notation being self-explanatory), which implies $\rho(U_x\psi) = U^{(1)}_y \rho(\psi) U^{(1)}_{-y}$. For every $g \in G_1$

$$\operatorname{Tr}\bigl(\rho(U_x\psi)V^{(1)}_g\bigr) = \operatorname{Tr}\bigl(U^{(1)}_y \rho(\psi) U^{(1)}_{-y} V^{(1)}_g\bigr) = \operatorname{Tr}\bigl(\rho(\psi) U^{(1)}_{-y} V^{(1)}_g U^{(1)}_y\bigr) \,;$$

however, $V^{(1)}_g U^{(1)}_y = \exp(i\langle g, y\rangle) U^{(1)}_y V^{(1)}_g$ and $\langle g, y\rangle = \langle g, x\rangle$ (since $g \in (G^2)^\perp$); we have $\operatorname{Tr}\bigl(\rho(U_x\psi)V^{(1)}_g\bigr) = \exp(i\langle g, x\rangle) \operatorname{Tr}\bigl(\rho(\psi)V^{(1)}_g\bigr)$. Note that $|\operatorname{Tr}\bigl((\rho(U_x\psi) - \rho(\psi))V^{(1)}_g\bigr)| \le \|\rho(U_x\psi) - \rho(\psi)\|$, since $\operatorname{Re}\bigl(e^{i\alpha}\operatorname{Tr}\bigl((\rho(U_x\psi) - \rho(\psi))V^{(1)}_g\bigr) = \operatorname{Tr}\bigl((\rho(U_x\psi) - \rho(\psi))\operatorname{Re}(e^{i\alpha}V^{(1)}_g)\bigr) \le \|\rho(U_x\psi) - \rho(\psi)\|$ for all $\alpha$. We have

$$\begin{gathered}
\|\rho(U_x\psi) - \rho(\psi)\| \ge |1 - e^{i\langle g,x\rangle}| \cdot |\operatorname{Tr}\bigl(\rho(\psi)V^{(1)}_g\bigr)| \,; \\
|1 - e^{i\langle g,x\rangle}| = 2\Bigl|\sin\frac{\langle g, x\rangle}{2}\Bigr| \,; \\
\operatorname{Tr}\bigl(\rho(\psi)V^{(1)}_g\bigr) = \operatorname{Tr}\bigl((V^{(1)}_g \otimes \mathbf{1}_{H_2})Q_\psi\bigr) = \langle (V^{(1)}_g \otimes \mathbf{1}_{H_2})\psi, \psi\rangle = \\
= \langle V_g\psi, \psi\rangle = \exp\bigl(-\tfrac12\|g\|^2\bigr) \,;
\end{gathered}$$

so,

$$\|\rho(U_x\psi) - \rho(\psi)\| \ge \sup_{g\in G_1}\biggl(\exp\bigl(-\tfrac12\|g\|^2\bigr) \cdot 2\Bigl|\sin\frac{\langle g, x\rangle}{2}\Bigr|\biggr) .$$

Denote $r = \operatorname{dist}(x, G^2)$. We need only one ray of vectors $g \in G_1$ such that $\langle g, x\rangle = r\|g\|$. For every $\varphi \in [0, \pi]$ there exists such $g$, satisfying $\langle g, x\rangle = \varphi$ and $\|g\| = \varphi/r$, which gives $\|\rho(U_x\psi) - \rho(\psi)\| \ge \exp\bigl(-\frac{\varphi^2}{2r^2}\bigr) \cdot 2\sin\frac{\varphi}{2}$; the supremum over $\varphi$ gives the first inequality.

For the second inequality, the proof is quite similar. Only $U^{(1)}_y$ and $V^{(1)}_g$ change places, and $\exp\bigl(-\frac18\|x\|^2\bigr)$ appears instead of $\exp\bigl(-\frac12\|g\|^2\bigr)$, which leads to $M(2r)$ instead of $M(r)$. ∎

**5.3. Theorem.** Let $G_0$ be an FHS-space, $E_n, F_n \subset G_0$ subspaces such that $G_0 = E_n \oplus F_n$ (in the FHS sense) for all $n$. For every unit vector $\psi \in \operatorname{Exp}(G_0)$ let $\rho_n(\psi)$ denote the density matrix on $F_n$ that corresponds to $\psi$. Then the following two conditions are equivalent.

(a) $\liminf E_n = G_0$ and $\limsup F_n = \{0\}$.

(b) $\|\rho_n(\psi_1) - \rho_n(\psi_2)\| \to 0$ when $n \to \infty$, for all unit vectors $\psi_1, \psi_2 \in \operatorname{Exp}(G_0)$.

*Proof.* Proposition 5.1 gives (a) $\Longrightarrow$ (b). Assume (b); we have to prove (a). We choose a Gaussian measure $\gamma$ and apply Lemma 5.2 to $\psi = \sqrt{\gamma}$:

$$M\big(\operatorname{dist}(x, F_n^\perp)\big) \le \|\rho_n(\psi) - \rho_n(U_x\psi)\| \to 0\,,$$
$$M\big(2\operatorname{dist}(g, E_n)\big) \le \|\rho_n(\psi) - \rho_n(V_g\psi)\| \to 0\,,$$

which implies that $\operatorname{dist}(x, F_n^\perp) \to 0$ and $\operatorname{dist}(g, E_n) \to 0$ (when $n \to \infty$) for all $g \in G_0$, $x \in G^0$ (the dual to $G_0$). The latter, $\inf_{e \in E_n} \|g - e\| \to 0$, shows that $\liminf E_n = G_0$. The former, $\sup_{f \in F_n, \|f\| \le 1} \langle f, x \rangle \to 0$, shows that $\limsup F_n = \{0\}$. $\square$

# 6 Borel measurability of it all

Recall some notions and results about Borel measurability (see [5, Chapter 3], [11, Chapter 3]). A *Borel space* is a set equipped with a $\sigma$-field (of subsets). The subsets belonging to the $\sigma$-field are called Borel measurable sets (or 'measurable sets', or 'Borel sets'). A subset of a Borel space is naturally a Borel space. The product of two Borel spaces is naturally a Borel space. A Borel measurable map (or 'measurable map', or 'Borel map') is a map from one Borel space into another, such that the inverse image of every measurable set is a measurable set. A Borel isomorphism between two Borel spaces is an invertible measurable map whose inverse is also measurable. (Note that no measure (type) is given, and so, no subset is negligible; every single point counts.)

A *Polish space* is a topological space which is homeomorphic to a separable complete metric space. A Polish space is naturally equipped with the $\sigma$-field generated by all open sets, thus, it is a Borel space. Surprisingly, the Borel space does not depend (up to isomorphism) on the Polish space, as far as it is uncountable. That is, every two uncountable Polish spaces are Borel isomorphic. Moreover, all uncountable Borel sets in Polish spaces are Borel isomorphic. A Borel space is called *standard,* if it is isomorphic to a Borel subset of a Polish space. Up to isomorphism, there is a single uncountable standard Borel space, a single countable (infinite) one, and for each (finite) $n$, a single $n$-point one.

Let $X$ be a Polish space and $\mathbf{F}(X)$ the set of all nonempty closed subsets of $X$. There is a natural Borel structure on $\mathbf{F}(X)$, namely, the $\sigma$-field generated by sets of the form

$$\{F \in \mathbf{F}(X) : F \cap U \ne \emptyset\}$$

where $U$ varies over open sets in $X$. Thus $\mathbf{F}(X)$ is a Borel space; it is called the Effros Borel space of $X$, and is standard (see [11, Th. 3.3.10]). There is a sequence $(f_n)$ of Borel measurable maps $f_n : \mathbf{F}(X) \to X$ such that every $F \in \mathbf{F}(X)$ is the closure of the countable (or finite) set $\{f_1(F), f_2(F), \dots\}$; it is called Castaing's theorem (see [11, Prop. 5.2.7]). Therefore, for any Borel space $T$, a general form of a measurable map $f : T \to \mathbf{F}(X)$ is

$$f(t) = \operatorname{closure}\bigl(\{f_1(t), f_2(t), \dots\}\bigr) , \tag{6.1}$$

where $f_1, f_2, \dots : T \to X$ are measurable maps. Note that the disjoint union of all $F \in \mathbf{F}(X)$, defined as the set of all pairs $(F, x)$ such that $F \in \mathbf{F}(X)$ and $x \in F$, is naturally a standard Borel space, since it is a Borel subset of $\mathbf{F}(X) \times X$.

Now we apply all that to our matter. Let $H$ be a (separable) Hilbert space and $\mathbf{L}(H)$ the set of all (closed) linear subspaces of $H$. Then $H$ is a Polish space, $\mathbf{L}(H) \subset \mathbf{F}(H)$, and we get measurable maps $f_n : \mathbf{L}(H) \to H$ such that every $L \in \mathbf{L}(H)$ is spanned by (and even the closure of) $\{f_1(L), f_2(L), \dots\}$. Applying the usual orthogonalization process to the sequence $\bigl(f_n(L)\bigr)$ we get a new sequence, denote it again by $\bigl(f_n(L)\bigr)$, such that

$$\{f_n(L) : 1 \le n < 1 + \dim L\} \quad \text{is an orthonormal basis of } L \tag{6.2}$$

for every $L \in \mathbf{L}(H)$, and still, $f_n : \mathbf{L}(H) \to H$ are Borel measurable, and in addition, $f_n(L) = 0$ when $n \ge 1 + \dim L$. The same argument shows also that $\dim L \in \{0, 1, 2, \dots; \infty\}$ is a measurable function of $L$.

There is a natural map, $F \mapsto \operatorname{span} F$, from $\mathbf{F}(H)$ to $\mathbf{L}(H)$; namely, $\operatorname{span} F$ is the (closed) subspace spanned by $F$. The map is measurable, which follows from (6.1). Indeed, if $F$ is the closure of $\{f_n(F) : n = 1, 2, \dots\}$, then $\operatorname{span} F$ is the closure of

$$\{\alpha_1 f_1(F) + \dots + \alpha_n f_n(F) : \alpha_1, \dots, \alpha_n \in \mathbb{Q},\ n = 1, 2, \dots\} , \tag{6.3}$$

still a countable set of measurable functions (indexed by finite sequences $(\alpha_1, \dots, \alpha_n)$ of rational numbers).

**6.4. Lemma.** Linear subspaces of a Hilbert space are a standard Borel space.[19]

*Proof.* $\mathbf{L}(H) = \{F \in \mathbf{F}(H) : \operatorname{span}(F) = F\}$, therefore $\mathbf{L}(H)$ is a Borel subset of the standard Borel space $\mathbf{F}(H)$.[20] $\square$

[19] *Closed* linear subspaces of a *separable* Hilbert space, of course.

[20] Indeed, $F \mapsto (F, \operatorname{span} F)$ is a Borel map $\mathbf{F}(H) \to \mathbf{F}(H) \times \mathbf{F}(H)$, and the diagonal is a Borel subset of $\mathbf{F}(H) \times \mathbf{F}(H)$.

**6.5. Lemma.** The closure of $L_1 + L_2$ is a jointly Borel measurable function of linear subspaces $L_1, L_2$ of a Hilbert space.

The proof is left to the reader. Hint: Similar to (6.3) but simpler.[21]

**6.6. Lemma.** The orthogonal projection of $x$ to $L$ is jointly measurable in $x \in H$ and $L \in \mathbf{L}(H)$.

*Proof.* The projection is the limit (for $n \to \infty$) of the orthogonal projection of $x$ to $\operatorname{span}\{f_k(L) : k \le n\}$. Measurability of the latter implies that of the former. $\square$

Note also that the disjoint union of all $L \in \mathbf{L}(H)$ is a standard Borel space, and linear operations are measurable in the following sense: $(L, h_1 + h_2)$ is jointly Borel measurable in $(L, h_1)$ and $(L, h_2)$ on the domain consisting of all pairs $\big((L_1, h_1), (L_2, h_2)\big)$ where $L_1, L_2 \in \mathbf{L}(X)$, $h_1 \in L_1$, $h_2 \in L_2$ satisfy $L_1 = L_2$. The same for other linear combinations, and for the scalar product.

Let $H$ be an FHS-space (rather than a Hilbert space). Lemmas 6.4 and 6.5 still hold.

**6.7. Lemma.** The set of all pairs $(L_1, L_2)$ such that $H = L_1 \oplus L_2$ (in the FHS sense, see Sect. 3) is a Borel subset of $\mathbf{L}(H) \times \mathbf{L}(H)$.

*Proof.* The relation $H = L_1 \oplus L_2$ means that, first, $L_1, L_2$ are orthogonal in some admissible norm, and second, $L_1 + L_2$ is dense in $H$. The latter condition defines a measurable set of pairs $(L_1, L_2)$ due to Lemma 6.5. The former condition may be expressed in terms of the infinite matrix

$$M(L_1, L_2) = \big(\langle f_k(L_1), f_l(L_2)\rangle\big)_{k,l}$$

where $f_n$ are as in (6.2). The relevant set of matrices is Borel measurable. $\square$

We turn to $\sigma$-fields. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a measure type space and $\mathbf{A}(\mathcal{F})$ the set of all sub-$\sigma$-fields of $\mathcal{F}$.[22] Then $\mathcal{F}_0 = \mathcal{F} \bmod 0$ is a complete Boolean algebra and a Polish space (recall Footnote 8), and $\mathbf{A}(\mathcal{F})$ may be identified with the set $\mathbf{A}(\mathcal{F}_0)$ of all closed subalgebras of $\mathcal{F}_0$. Thus, $\mathbf{A}(\mathcal{F}) = \mathbf{A}(\mathcal{F}_0) \subset \mathbf{F}(\mathcal{F}_0)$, and we get measurable maps $f_n : \mathbf{A}(\mathcal{F}_0) \to \mathcal{F}_0$ such that every $\mathcal{A} \in \mathbf{A}(\mathcal{F}_0)$ is generated by (and even the closure of) $\{f_1(\mathcal{A}), f_2(\mathcal{A}), \dots\}$.

Striving to a counterpart of (6.2), recall the space $L_0(\Omega, \mathcal{F}, \mathcal{P})$ of all equivalence classes of measurable maps $\Omega \to \mathbb{R}$. Every $X \in L_0(\Omega, \mathcal{F}, \mathcal{P})$

[21] An alternative way: $L_1 \cup L_2$ is measurable in $L_1, L_2$ by [11, Exercise 3.3.11(ii)], thus $\operatorname{span}(L_1 \cup L_2)$ is also measurable. It is a good luck that we need unions, not intersections; see the note after Exercise 3.3.11 in [11].

[22] As before, each $\sigma$-field must contain all $\mathcal{P}$-negligible sets.

generates a $\sigma$-field $\sigma(X) \in \mathbf{A}(\mathcal{F}_0)$. Every $\mathcal{A} \in \mathbf{A}(\mathcal{F}_0)$ is $\sigma(X)$ for some $X \in L_0(\Omega, \mathcal{F}, \mathcal{P})$; the set of all such $X$ (for a given $\mathcal{A}$) is usually large and non-closed. Nevertheless a selection is constructed below.

**6.8. Lemma.** There exists a Borel map $\mathcal{A} \to X_{\mathcal{A}}$ from $\mathbf{A}(\mathcal{F}_0)$ to $L_0(\Omega, \mathcal{F}, \mathcal{P})$ such that $\sigma(X_{\mathcal{A}}) = \mathcal{A}$ for all $A \in \mathbf{A}(\mathcal{F}_0)$.

*Proof.* One may take

$$X_{\mathcal{A}}(\omega) = \sum_{n=1}^{\infty} \frac{2}{3^n} \mathbf{1}_{f_n(\mathcal{A})}(\omega) \, ;$$

here $\mathbf{1}_{f_n(\mathcal{A})}(\omega)$ is equal to 1 if $\omega \in f_n(\mathcal{A})$ and 0 otherwise. $\square$

A $\sigma$-field $\mathcal{A}$ is nonatomic if and only if $X_{\mathcal{A}}$ is nonatomic (that is, $X_{\mathcal{A}}^{-1}(\{x\})$ is negligible for every $x \in \mathbb{R}$). Nonatomic elements of $L_0(\Omega, \mathcal{F}, \mathcal{P})$ are a Borel set. Therefore, nonatomic $\sigma$-fields are a Borel set (in $\mathbf{A}(\mathcal{F}_0)$). Similarly, the number of atoms ($0, 1, 2, \dots$ or $\infty$) is a Borel function of $\mathcal{A}$, as well as their (ordered) probabilities.

There is a natural map, $F \mapsto \sigma(F)$, from $\mathbf{F}(\mathcal{F}_0)$ to $\mathbf{A}(\mathcal{F}_0)$; namely, $\sigma(F)$ is the $\sigma$-field generated by $F$. The map is measurable, which follows from (6.1) similarly to (6.3).[23]

Proofs of the following Lemmas 6.9–6.11 are left to the reader, since they are similar to 6.4–6.6 respectively. Especially for 6.11 a hint: $\dfrac{Q|_{\mathcal{A}}}{P|_{\mathcal{A}}} = \lim\limits_{n\to\infty} \dfrac{Q|_{\mathcal{A}_n}}{P|_{\mathcal{A}_n}}$ where $\mathcal{A}_n$ is generated by $f_1(\mathcal{A}), \dots, f_n(\mathcal{A})$.

**6.9. Lemma.** Sub-$\sigma$-fields of $\mathcal{F}$ are a standard Borel space.

**6.10. Lemma.** The $\sigma$-field $\sigma(\mathcal{A}_1 \cup \mathcal{A}_2)$ generated by $\mathcal{A}_1, \mathcal{A}_2$ is a jointly measurable function of sub-$\sigma$-fields $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{F}$.

The set $\mathcal{P}$ of equivalent probability measures on $(\Omega, \mathcal{F})$ is also a standard Borel space, since it is homeomorphic (in fact, isometric) to a subset of $L_1(P)$ (the choice of $P \in \mathcal{P}$ does not matter); the whole $L_1(P)$ is a Polish space, and the subset, consisting of all strictly positive functions whose integral is equal to 1, is a Borel subset.

[23] Arbitrary combinations of Boolean operations (union, intersection, complement) are used here instead of the linear combinations used in (6.2).

**6.11. Lemma.** The Radon-Nikodym density

$$\frac{Q|_{\mathcal{A}}}{P|_{\mathcal{A}}}$$

(treated as an element of $L_2(\Omega, \mathcal{F}, \mathcal{P})$ that belongs in fact to $L_0(\Omega, \mathcal{A}, \mathcal{P})$) is jointly measurable in $P, Q \in \mathcal{P}$ and $\mathcal{A} \in \mathbf{A}(\mathcal{F})$.

Given $P \in \mathcal{P}$ and $X \in L_0(\Omega, \mathcal{F}, \mathcal{P})$, we get a probability measure on $\mathbb{R}$, namely, $S \mapsto P\bigl(\{\omega : X(\omega) \in S\}\bigr)$ for Borel sets $S \subset \mathbb{R}$. The measure will be called the distribution of $X$ w.r.t. $P$ and denoted by $X(P)$. Note that $X(P)$ is jointly measurable in $X$ and $P$, that is, $(X, P) \mapsto X(P)$ is a Borel map from $L_0(\mathcal{P}) \times \mathcal{P}$ to the space of probability distributions on $\mathbb{R}$. Indeed, if $\varphi$ is a bounded continuous function $\mathbb{R} \to \mathbb{R}$ then $\int \varphi \, d(X(P)) = \int \varphi(X(\cdot)) \, dP$ is continuous in $(X, P)$.

**6.12. Lemma.** (a) The set $\mathcal{D}$ of all pairs $(\mathcal{A}_1, \mathcal{A}_2)$ such that $\mathcal{A}_1 \otimes \mathcal{A}_2$ is well-defined,[24] is a Borel subset of $\mathbf{A}(\mathcal{F}) \times \mathbf{A}(\mathcal{F})$, and the map $(\mathcal{A}_1, \mathcal{A}_2) \mapsto \mathcal{A}_1 \otimes \mathcal{A}_2$ from $\mathcal{D}$ to $\mathbf{A}(\mathcal{F})$ is Borel measurable.

(b) For every $P \in \mathcal{P}$, the map

$$(\mathcal{A}_1, \mathcal{A}_2) \mapsto \frac{P|_{\mathcal{A}_1 \otimes \mathcal{A}_2}}{P|_{\mathcal{A}_1} \otimes P|_{\mathcal{A}_2}}$$

from $\mathcal{D}$ to $L_0(\mathcal{P})$ is Borel measurable.

*Proof.* (a) The condition $(\mathcal{A}_1, \mathcal{A}_2) \in \mathcal{D}$ may be expressed in terms of the measure

$$\mu_{\mathcal{A}_1, \mathcal{A}_2} = (X_{\mathcal{A}_1}, X_{\mathcal{A}_2})(P)$$

on $\mathbb{R}^2$. (The choice of $P \in \mathcal{P}$ does not matter.) That is the joint probability distribution of random variables $X_{\mathcal{A}_1}, X_{\mathcal{A}_2}$ on $(\Omega, \mathcal{F}, P)$, and its marginal distributions are $X_{\mathcal{A}_1}(P), X_{\mathcal{A}_2}(P)$. Clearly, $\mu_{\mathcal{A}_1, \mathcal{A}_2}$ is a product measure if and only if $\mathcal{A}_1, \mathcal{A}_2$ are independent w.r.t. $P$. Thus, the relevant condition on $\mu_{\mathcal{A}_1, \mathcal{A}_2}$ says that $\mu_{\mathcal{A}_1, \mathcal{A}_2}$ must be equivalent to a product measure. The set of all such measures on $\mathbb{R}^2$ is Borel measurable. The map $(\mathcal{A}_1, \mathcal{A}_2) \mapsto \mathcal{A}_1 \otimes \mathcal{A}_2$ is the restriction to $\mathcal{D}$ of the map $(\mathcal{A}_1, \mathcal{A}_2) \mapsto \sigma(\mathcal{A}_1 \cup \mathcal{A}_2)$ measurable by Lemma 6.10.

[24] It means existence of $P \in \mathcal{P}$ that makes $\mathcal{A}_1, \mathcal{A}_2$ independent (recall Def. 3.3). When defined, $\mathcal{A}_1 \otimes \mathcal{A}_2$ is just $\sigma(\mathcal{A}_1 \cup \mathcal{A}_2)$.

(b) For such $\mathcal{A}_1, \mathcal{A}_2$ the map $(X_{\mathcal{A}_1}, X_{\mathcal{A}_2})$ is an isomorphism (mod0) between $(\Omega, \mathcal{A}_1 \otimes \mathcal{A}_2, P)$ and $(\mathbb{R}^2, \dots, \mu)$, where $\mu = \mu_{\mathcal{A}_1, \mathcal{A}_2}$ (and the $\sigma$-field of $\mu$-measurable subsets of $\mathbb{R}^2$ is suppressed in the notation). Therefore,

$$\frac{P}{P|_{\mathcal{A}_1} \otimes P|_{\mathcal{A}_2}}(\omega) = \frac{\mu}{\mu_1 \otimes \mu_2}\big(X_{\mathcal{A}_1}(\omega), X_{\mathcal{A}_2}(\omega)\big) \,,$$

where $\mu_1 = \mu_{\mathcal{A}_1} = X_{\mathcal{A}_1}(P)$, $\mu_2 = \mu_{\mathcal{A}_2} = X_{\mathcal{A}_2}(P)$ are the marginals of $\mu$.

I assume in addition that the $\sigma$-fields $\mathcal{A}_1, \mathcal{A}_2$ are nonatomic (atoms are left to the reader); then an additional transformation $\mathbb{R} \to \mathbb{R}$ turns $\mu_1, \mu_2$ into Lebesgue measure on $(0,1)$.[25] The density of $\mu$ (w.r.t. Lebesgue measure $\mu_1 \otimes \mu_2$ on $(0,1) \times (0,1)$) is a Borel function of $\mu$ (take an increasing (refining) sequence of finite partitions of $(0,1) \times (0,1)$). The following lemma (or rather, its straightforward two-dimensional generalization) completes the proof. $\square$

**6.13. Lemma.** Let $(\Omega, \mathcal{F}, P)$ be a nonatomic probability space. Consider the set $\mathcal{U} \subset L_0(\Omega, \mathcal{F}, P)$ of all random variables $U : \Omega \to \mathbb{R}$ distributed uniformly on $(0,1)$ (in other words, measure preserving transformations from $(\Omega, \mathcal{F}, P)$ to $(0,1)$ with Lebesgue measure). For each $U \in \mathcal{U}$ and $f \in L_0(0,1)$ consider the composition $f \circ U$ (that is, $f(U(\cdot))$) as an element of $L_0(\Omega, \mathcal{F}, P)$. Then

(a) $U$ is a Borel measurable subset of $L_0(\Omega, \mathcal{F}, P)$;

(b) $f \circ U$ is jointly Borel measurable in $f \in L_0(0,1)$ and $U \in \mathcal{U}$.

*Proof.* (a) Follows immediately from measurability of $U(P)$ in $U$.

(b) Let $Q$ be another probability measure on $(\Omega, \mathcal{F})$, equivalent to $P$. Consider the distribution

$$(f \circ U)(Q) = f(U(Q)) \,;$$

we know that $f(\mu)$ is jointly measurable in $f$ and $\mu$, and $U(Q)$ is measurable in $U$, therefore $(f \circ U)(Q)$ is jointly measurable in $f$ and $U$. However, distributions $X(Q)$ for all $Q$ determine $X \in L_0(P)$ uniquely, and moreover, they generate the Borel $\sigma$-field on $L_0(P)$. $\square$

**6.14. Proposition.** Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a measure type space, and $\mathcal{H} = \{(\mathcal{A}, \psi) : \mathcal{A} \in \mathbf{A}(\mathcal{F}), \psi \in L_2(\Omega, \mathcal{A}, \mathcal{P})\}$ the disjoint union of Hilbert spaces $L_2(\Omega, \mathcal{A}, \mathcal{P})$ over all sub-$\sigma$-fields $\mathcal{A} \subset \mathcal{F}$. Then

[25] One may use the cumulative distribution function $F_{\mathcal{A}}(x) = P\big(\{\omega : X_{\mathcal{A}}(\omega) \le x\}\big)$; it is continuous (due to the nonatomicity), and the random variable $\omega \mapsto F_{\mathcal{A}}(X_{\mathcal{A}}(\omega))$ is distributed uniformly on $(0,1)$.

(a) $\mathcal{H}$ is (naturally) a standard Borel space;

(b) the set $\mathcal{D}_1$ of all pairs $\big((\mathcal{A}_1,\psi_1),(\mathcal{A}_2,\psi_2)\big) \in \mathcal{H}\times\mathcal{H}$ satisfying $\mathcal{A}_1 = \mathcal{A}_2$ is a Borel subset of $\mathcal{H}\times\mathcal{H}$; the map $\big((\mathcal{A},\psi_1),(\mathcal{A},\psi_2)\big) \mapsto (\mathcal{A},\psi_1+\psi_2)$ from $\mathcal{D}_1$ to $\mathcal{H}$ is Borel measurable; the map $\big(\alpha,(\mathcal{A},\psi)\big) \mapsto (\mathcal{A},\alpha\psi)$ from $\mathbb{R}\times\mathcal{H}$ to $\mathcal{H}$ is Borel measurable; and the map $\big((\mathcal{A},\psi_1),(\mathcal{A},\psi_2)\big) \mapsto \langle\psi_1,\psi_2\rangle$ from $\mathcal{D}_1$ to $\mathbb{R}$ is Borel measurable;

(c) the set $\mathcal{D}_2$ of all pairs $\big((\mathcal{A}_1,\psi_1),(\mathcal{A}_2,\psi_2)\big) \in \mathcal{H}\times\mathcal{H}$ such that $\mathcal{A}_1\otimes\mathcal{A}_2$ is well-defined[26] is a Borel subset of $\mathcal{H}\times\mathcal{H}$; and the map $\big((\mathcal{A}_1,\psi_1),(\mathcal{A}_2,\psi_2)\big) \mapsto \big(\mathcal{A}_1\otimes\mathcal{A}_2,\psi_1\otimes\psi_2\big)$ from $\mathcal{D}_2$ to $\mathcal{H}$ is Borel measurable.

*Proof.* We choose some $P\in\mathcal{P}$ and replace each $L_2(\Omega,\mathcal{A},\mathcal{P})$ with the corresponding $L_2(\Omega,\mathcal{A},P)$ according to their unitary correspondence determined by $P$,

$$L_2(\Omega,\mathcal{A},\mathcal{P}) \ni \psi \mapsto \frac{\psi}{\sqrt{P|_\mathcal{A}}} \in L_2(\Omega,\mathcal{A},P)\,.$$

The disjoint union of $L_2(\Omega,\mathcal{A},P)$ over all $\mathcal{A}$ is naturally a standard Borel space, since it is a Borel subset of the disjoint union of all subspaces of $L_2(\Omega,\mathcal{F},P)$.[27] Thus, a Borel structure appears also on the disjoint union of $L_2(\Omega,\mathcal{A},\mathcal{P})$, and the Borel structure does not depend on the choice of $P$ (since $\dfrac{Q|_\mathcal{A}}{P|_\mathcal{A}}$ is measurable in $\mathcal{A}$, see Lemma 6.11). Item (b) is easily transferred from the disjoint union of all subspaces of $L_2(\Omega,\mathcal{F},P)$ to the disjoint union of $L_2(\Omega,\mathcal{A},\mathcal{P})$. It remains to prove (c).

Lemma 6.12(a) gives us measurability of the set $\mathcal{D}_2$, and measurability of $\mathcal{A}_1\otimes\mathcal{A}_2$ in $(\mathcal{A}_1,\mathcal{A}_2)$. It remains to verify measurability of $\psi_1\otimes\psi_2$. We know (recall Sect. 4) that

$$\psi_1\otimes\psi_2 = \bigg(\frac{\psi_1}{\sqrt{Q_1}}\otimes\frac{\psi_2}{\sqrt{Q_2}}\bigg)\cdot\sqrt{Q_1\otimes Q_2} \in L_2(\Omega,\mathcal{A}_1\otimes\mathcal{A}_2,\mathcal{P})\,;$$

that is, if $Q\in\mathcal{P}$ is a measure that makes $\mathcal{A}_1,\mathcal{A}_2$ independent, then

$$\frac{\psi_1\otimes\psi_2}{\sqrt{Q|_{\mathcal{A}_1\otimes\mathcal{A}_2}}} = \frac{\psi_1}{\sqrt{Q|_{\mathcal{A}_1}}}\cdot\frac{\psi_2}{\sqrt{Q|_{\mathcal{A}_2}}} \quad \text{for } \psi_1\in L_2(\Omega,\mathcal{A}_1,\mathcal{P}),\ \psi_2\in L_2(\Omega,\mathcal{A}_2,\mathcal{P})\,;$$

[26] See Lemma 6.12.

[27] You see, $L_2(\Omega,\mathcal{A},\mathcal{P})$ is not (naturally identified with) a subspace of $L_2(\Omega,\mathcal{F},\mathcal{P})$. However, $L_2(\Omega,\mathcal{A},P)$ is a subspace of $L_2(\Omega,\mathcal{F},P)$. Thus, all $L_2(\Omega,\mathcal{A},\mathcal{P})$ become embedded into $L_2(\Omega,\mathcal{F},\mathcal{P})$, but the embedding depends on $P$. It may be written as $L_2(\Omega,\mathcal{A},\mathcal{P}) \ni \psi \mapsto \dfrac{\psi}{\sqrt{P|_\mathcal{A}}}\cdot\sqrt{P} \in L_2(\Omega,\mathcal{F},P)$.

the product in the right-hand side is just a pointwise product of two functions.[28] Therefore

$$\frac{\psi_1 \otimes \psi_2}{\sqrt{P|_{\mathcal{A}_1 \otimes \mathcal{A}_2}}} = \sqrt{\frac{Q|_{\mathcal{A}_1 \otimes \mathcal{A}_2}}{P|_{\mathcal{A}_1 \otimes \mathcal{A}_2}}} \cdot \sqrt{\frac{P|_{\mathcal{A}_1}}{Q|_{\mathcal{A}_1}}} \cdot \sqrt{\frac{P|_{\mathcal{A}_2}}{Q|_{\mathcal{A}_2}}} \cdot \frac{\psi_1}{\sqrt{P|_{\mathcal{A}_1}}} \cdot \frac{\psi_2}{\sqrt{P|_{\mathcal{A}_2}}} .$$

However, the independence of $\mathcal{A}_1$ and $\mathcal{A}_2$ under $Q$ means that

$$\frac{P|_{\mathcal{A}_1 \otimes \mathcal{A}_2}}{Q|_{\mathcal{A}_1 \otimes \mathcal{A}_2}} = \frac{P|_{\mathcal{A}_1 \otimes \mathcal{A}_2}}{Q|_{\mathcal{A}_1} \otimes Q|_{\mathcal{A}_2}} = \frac{P|_{\mathcal{A}_1 \otimes \mathcal{A}_2}}{P|_{\mathcal{A}_1} \otimes P|_{\mathcal{A}_2}} \cdot \frac{P|_{\mathcal{A}_1}}{Q|_{\mathcal{A}_1}} \cdot \frac{P|_{\mathcal{A}_2}}{Q|_{\mathcal{A}_2}} ,$$

so,

$$\frac{\psi_1 \otimes \psi_2}{\sqrt{P|_{\mathcal{A}_1 \otimes \mathcal{A}_2}}} = \sqrt{\frac{P|_{\mathcal{A}_1} \otimes P|_{\mathcal{A}_2}}{P|_{\mathcal{A}_1 \otimes \mathcal{A}_2}}} \cdot \frac{\psi_1}{\sqrt{P|_{\mathcal{A}_1}}} \cdot \frac{\psi_2}{\sqrt{P|_{\mathcal{A}_2}}} .$$

By Lemma 6.12(b), $\dfrac{P|_{\mathcal{A}_1 \otimes \mathcal{A}_2}}{P|_{\mathcal{A}_1} \otimes P|_{\mathcal{A}_2}}$ is Borel measurable in $(\mathcal{A}_1, \mathcal{A}_2)$, therefore $\dfrac{\psi_1 \otimes \psi_2}{\sqrt{P|_{\mathcal{A}_1 \otimes \mathcal{A}_2}}}$ is measurable in $\left(\mathcal{A}_1, \mathcal{A}_2, \dfrac{\psi_1}{\sqrt{P|_{\mathcal{A}_1}}}, \dfrac{\psi_2}{\sqrt{P|_{\mathcal{A}_2}}}\right)$, which means that $\psi_1 \otimes \psi_2$ is measurable in $\big((\mathcal{A}_1, \psi_1), (\mathcal{A}_2, \psi_2)\big)$. □

For any Hilbert spaces $H_1, H_2$ denote by $\tilde{\mathcal{I}}(H_1, H_2)$ the set of all isomorphisms between $H_1$ and $H_2$, that is, linear isometric invertible maps $H_1 \to H_2$ (of course, $\tilde{\mathcal{I}}(H_1, H_2)$ is empty if $H_1, H_2$ are of different dimension). For any Hilbert space $H$ denote by $\mathcal{I}(H)$ the disjoint union of sets $\tilde{\mathcal{I}}(L_1, L_2)$ over all subspaces $L_1, L_2 \in \mathbf{L}(H)$. That is, $\mathcal{I}(H)$ consists of all triples $(L_1, L_2, U)$ where $L_1, L_2 \subset H$ are subspaces and $U : L_1 \to L_2$ is an isomorphism. However, we may identify each $U$ with its graph, and $L_1, L_2$ with projections of the graph, $L_1(U) = \{x : (x, y) \in U\}$, $L_2(U) = \{y : (x, y) \in U\}$. Now $\mathcal{I}(H)$ consists of all subspaces $U \in \mathbf{L}(H \oplus H)$ such that $\|x\| = \|y\|$ whenever $x \in H$, $y \in H$, $(x, y) \in U$. Clearly, $\mathcal{I}(H)$ is a Borel subset of $\mathbf{L}(H \oplus H)$, therefore, a standard Borel space.

**6.15. Lemma.** Let $H$ be a Hilbert space. Then

(a) the set $\mathcal{D}$ of all pairs $\big((L_1, L_2, U), x\big)$ such that $(L_1, L_2, U) \in \mathcal{I}(H)$ and $x \in L_1$ is a Borel subset of $\mathcal{I}(H) \times H$;

(b) the map $\big((L_1, L_2, U), x\big) \mapsto U(x)$ from $\mathcal{D}$ to $H$ is Borel measurable.

*Proof.* Treating $U$ as a subspace of $H \oplus H$ we choose measurable maps $f_n : \mathbf{L}(H \oplus H) \to H \oplus H$ such that every $U$ is spanned by $\{f_1(U), f_2(U), \dots\}$.

[28] It is, at the same time, their tensor product, since they are independent (w.r.t. $Q$).

We have $f_n(U) = (g_n(U), h_h(U))$ where $g_n(U) \in H$, $h_n(U) \in H$. Applying the orthogonalization process we ensure that $g_n(U)$ form an orthogonal basis of $L_1(U)$. Introducing Borel functions $c_n(x, U) = \langle x, g_n(U) \rangle$ for $x \in H$ we have

$$(U, x) \in \mathcal{D} \quad \Longleftrightarrow \quad \sum_n |c_n(x, U)|^2 = \|x\|^2 \,,$$

$$(U, x) \in \mathcal{D} \quad \Longrightarrow \quad U(x) = \sum_n c_n(x, U) h_n(U) \,.$$

□

For any FHS space $G$ we define $\mathcal{I}(G)$ as consisting of all triples $(L_1, L_2, U)$ where $L_1, L_2 \subset G$ are subspaces and $U : L_1 \to L_2$ is an FHS-isomorphism. Alternatively, $\mathcal{I}(G)$ may be thought of as a subset of $\mathbf{L}(H \oplus H)$; we'll see (Lemma 6.18) that it is a Borel subset, therefore, a standard Borel space.

For any measure type space $(\Omega, \mathcal{F}, \mathcal{P})$ we define $\mathcal{I}(\mathcal{F})$ as consisting of all triples $(\mathcal{A}_1, \mathcal{A}_2, U)$ where $\mathcal{A}_1, \mathcal{A}_2 \in \mathbf{A}(\mathcal{F} \bmod 0)$ and $U : \mathcal{A}_1 \to \mathcal{A}_2$ is an isomorphism of complete Boolean algebras (it means mod0 isomorphism between quotient spaces $\Omega/\mathcal{A}_1$ and $\Omega/\mathcal{A}_2$, provided that $(\Omega, \mathcal{A}, \mathcal{P})$ is a Lebesgue-Rokhlin space). The graph of $U$ is a subset of $(\mathcal{F} \bmod 0) \oplus (\mathcal{F} \bmod 0) = (\mathcal{F} \oplus \mathcal{F}) \bmod 0$, where $\mathcal{F} \oplus \mathcal{F}$ is the natural $\sigma$-field on the union $\Omega \uplus \Omega$ of two disjoint copies of $\Omega$. We identify $U$ with its graph; it is not just a subset but a $\sigma$-field; so, $\mathcal{I}(\mathcal{F}) \subset \mathbf{A}(\mathcal{F} \oplus \mathcal{F})$.

**6.16. Lemma.** $\mathcal{I}(\mathcal{F})$ is a Borel subset of $\mathbf{A}(\mathcal{F} \oplus \mathcal{F})$.

*Proof.* A sub-$\sigma$-field $\mathcal{B} \subset \mathcal{F} \oplus \mathcal{F}$ belongs to $\mathcal{I}(\mathcal{F})$ if and only if $\forall \varepsilon \exists \delta \forall (A, B) \in \mathcal{B} \big( P(A) < \delta \quad \Longrightarrow \quad P(B) \le \varepsilon \big)$; here an element of $\mathcal{F} \oplus \mathcal{F}$ is identified with a pair $(A, B)$ of elements of $\mathcal{F}$. (The choice of a measure $P \in \mathcal{P}$ does not matter.) For such $\varepsilon$ and $\delta$, $\mathcal{B}$ must be disjoint to the *open* set $\{(A, B) : P(A) < \delta, P(B) > \varepsilon\}$. □

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a measure type space. For any closed set $F \subset L_0(\Omega, \mathcal{F}, \mathcal{P})$ denote by $\sigma(F)$ the sub-$\sigma$-field generated by $F$, that is, the least $\sigma$-field $\mathcal{F}' \subset \mathcal{F}$ such that $F \subset L_0(\Omega, \mathcal{F}', \mathcal{P})$.

**6.17. Lemma.** The map $F \mapsto \sigma(F)$ from $\mathbf{F}\big(L_0(\Omega, \mathcal{F}, \mathcal{P})\big)$ to $\mathbf{A}(\Omega, \mathcal{F}, \mathcal{P})$ is Borel measurable.

*Proof.* Due to (6.1) it suffices to prove measurability of the map $f \mapsto \sigma(f)$ from $L_0(\Omega, \mathcal{F}, \mathcal{P})$ to $\mathbf{A}(\Omega, \mathcal{F}, \mathcal{P})$; of course, $\sigma(f)$ means $\sigma(\{f\})$. We know (see the proof of Lemma 3.8) that $f_n \to f$ implies $\sigma(f) \subset \liminf \sigma(f_n)$. Therefore, for any open set $V \subset \mathbf{A}(\Omega, \mathcal{F}, \mathcal{P})$ the set of all $f$ such that $\sigma(f) \cap V = \emptyset$ is closed. □

**6.18. Lemma.** Let $G$ be an FHS space, then $\mathcal{I}(G)$ is a Borel subset of $\mathbf{L}(G \oplus G)$.

*Proof.* We take a Gaussian type space $(\Omega, \mathcal{F}, \mathcal{P}, G)$[29] and consider $\mathcal{F} \oplus \mathcal{F}$, so that $G \oplus G \subset L_0(\mathcal{F} \oplus \mathcal{F})$. Every subspace $U \in \mathbf{L}(G \oplus G)$ generates a sub-$\sigma$-field $\sigma(U) \in \mathbf{A}(\mathcal{F} \oplus \mathcal{F})$. It is easy to see that $U \in \mathcal{I}(G)$ if and only if $\sigma(U) \in \mathcal{I}(\mathcal{F})$. However, $\sigma(U)$ is measurable in $U$ by Lemma 6.17, and $\mathcal{I}(\mathcal{F})$ is measurable by Lemma 6.16. $\square$

For any measure type space $(\Omega, \mathcal{F}, \mathcal{P})$ we define $\mathcal{I}(\mathcal{P})$[30] as consisting of all triples $(\mathcal{A}_1, \mathcal{A}_2, U)$ where $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{F}$ are sub-$\sigma$-fields and $U \in \tilde{\mathcal{I}}\big(L_2(\Omega, \mathcal{A}_1, \mathcal{P}), L_2(\Omega, \mathcal{A}_2, \mathcal{P})\big)$ is a linear isometry. Spaces $L_2(\Omega, \mathcal{A}, \mathcal{P})$ are not naturally embedded into $L_2(\Omega, \mathcal{F}, \mathcal{P})$; however, we may choose some measure $P \in \mathcal{P}$ and embed all $L_2(\Omega, \mathcal{A}, \mathcal{P})$ into $L_2(\Omega, \mathcal{F}, P)$ by

$$L_2(\Omega, \mathcal{A}, \mathcal{P}) \ni \psi \mapsto \frac{\psi}{\sqrt{P|_{\mathcal{A}}}} \in L_2(\Omega, \mathcal{A}, P) \subset L_2(\Omega, \mathcal{F}, P) \, .$$

We have a bijective correspondence between $\mathcal{I}(\mathcal{P})$ and $\mathcal{I}\big(L_2(\Omega, \mathcal{F}, P)\big)$, which turns $\mathcal{I}(\mathcal{P})$ into a standard Borel space. Its Borel structure does not depend on the choice of $P \in \mathcal{P}$ (recall Lemma 6.11), but the correspondence depends on $P$.

Take an element $(\mathcal{A}_1, \mathcal{A}_2, U)$ of $\mathcal{I}(\mathcal{F})$ (this time we prefer a triple to a graph). The isomorphism $U$ between $\sigma$-fields $\mathcal{A}_1, \mathcal{A}_2$ induces naturally an isomorphism (linear isometry) between Hilbert spaces $L_2(\Omega, \mathcal{A}_1, \mathcal{P})$ and $L_2(\Omega, \mathcal{A}_2, \mathcal{P})$. Namely, if measures $P_1 \in \mathcal{P}|_{\mathcal{A}_1}$, $P_2 \in \mathcal{P}|_{\mathcal{A}_2}$ satisfy $P_2(U(A)) = P_1(A)$ for all $A \in \mathcal{A}_1$, then the vector $\psi_2 = \sqrt{P_2} \in L_2(\Omega, \mathcal{A}_2, \mathcal{P})$ corresponds to the vector $\psi_1 = \sqrt{P_1} \in L_2(\Omega, \mathcal{A}_1, \mathcal{P})$. (Such vectors are not a linear set, but span the Hilbert spaces.) So, we have a map from $\mathcal{I}(\mathcal{F})$ to $\mathcal{I}(\mathcal{P})$.

**6.19. Lemma.** The map from $\mathcal{I}(\mathcal{F})$ to $\mathcal{I}(\mathcal{P})$ is Borel measurable.

*Proof.* Consider some $U \in \mathcal{I}(\mathcal{F})$ treated as a sub-$\sigma$-field of $\mathcal{F} \oplus \mathcal{F}$. Given some $P_1 \in \mathcal{P}$, we introduce on $U$ a measure $P(A, B) = P_1(A)$ for $(A, B) \in U$; here, as before, an element of $\mathcal{F} \oplus \mathcal{F}$ is represented by a pair $(A, B)$ where $A, B \in \mathcal{F}$. There is also a measure $P_2$ on the $\sigma$-field $\mathcal{A}_2 = \{B : (A, B) \in U\}$ such that $P_2(B) = P(A, B) = P_1(A)$ whenever $(A, B) \in U$. The pair $\big(\sqrt{P_1}, \sqrt{P_2}\big)$ belongs to the graph $U_1 \in \mathcal{I}(\mathcal{P})$ that corresponds to $U$. Such pairs for all $P_1 \in \mathcal{P}$ (or for a countable dense subset) span the graph $U_1$. It remains to prove that, for a given $P_1$ and arbitrary $U$, the pair $\big(\sqrt{P_1}, \sqrt{P_2}\big)$

[29] Suppressing in the notation the distinction between $G$ and $G/\,\mathrm{Const}$.

[30] Sorry for the clumsy notation: $\mathcal{I}(\mathcal{F})$ for $\sigma$-fields, but $\mathcal{I}(\mathcal{P})$ for square roots of measures.

is measurable in $U$ (you see, $P_2$ depends implicitly on $U$). According to our definition of the Borel structure on $\mathcal{I}(\mathcal{P})$, we have to prove measurability in $U$ of the density $\frac{P_2}{P_1|_{\mathcal{A}_2}}$. The latter is the restriction (to the second copy of $\Omega$) of a density on the doubled space, $\Omega \uplus \Omega$, namely, $\frac{P'|_U}{P''|_U}$, where measures $P', P''$ on $\mathcal{F} \oplus \mathcal{F}$ are defined (irrespective of $U$) by $P'(A, B) = P_1(A)$, $P''(A, B) = P_1(B)$. We apply Lemma 6.11 to $P', P''$ on $\Omega \uplus \Omega$. Though, $P'$ and $P''$ are not equivalent, but one can consider, say, $\frac{2P' + P''}{P' + P''}$. □

**6.20. Proposition.** Let $(\Omega, \mathcal{F}, \mathcal{P}, G)$ be a Gaussian type space, and $G_0 = G/\operatorname{Const}$ the corresponding FHS space. Then the natural map from $\mathcal{I}(G_0)$ to $\mathcal{I}(\mathcal{P})$ is Borel measurable.

*Proof.* We know that every $U \in \mathcal{I}(G)$ generates $\sigma(U) \in \mathcal{I}(\mathcal{F})$ (see the proof of Lemma 6.18), and the map $U \mapsto \sigma(U)$ is measurable. The transition from $\sigma(U)$ to the corresponding element of $\mathcal{I}(\mathcal{P})$ is measurable by Lemma 6.19. □

# 7 Sum systems and product systems

**7.1. Definition.** A *sum system* consists of a two-parameter family $(G_{a,b})$ of FHS-spaces $G_{a,b}$, given for $-\infty < a < b < +\infty$, embedded into a single linear space $G_{-\infty,+\infty}$, and a one-parameter group $(U_t)$ of linear maps $U_t : G_{-\infty,+\infty} \to G_{-\infty,+\infty}$ for $t \in \mathbb{R}$, such that

(a) $G_{a,c} = G_{a,b} \oplus G_{b,c}$ (in the FHS sense) whenever $-\infty < a < b < c < +\infty$;[31]

(b) $U_t : G_{a,b} \to G_{a+t,b+t}$ is an isomorphism of FHS spaces, whenever $-\infty < a < b < +\infty$ and $-\infty < t < +\infty$.[32]

(c) $(U_t)$ is strongly continuous in the sense that $\|U_t x - x\| \to 0$ when $t \to 0$, whenever $a < b < c < d$, $x \in G_{b,c}$, and the norm is taken in $G_{a,b}$ (which is correct for $t$ small enough).

The structure is local in the sense that the global space $G_{-\infty,+\infty}$ is equipped with a linear structure only, not an FHS structure, nor even a

[31]Thus, $H_{a,b}$ and $H_{b,c}$ must be linear subspaces of $H_{a,c}$; and their FHS structures must be inherited from $H_{a,c}$; and they must be orthogonal in some admissible norm. Note that the norm may depend on $b$.

[32]Thus, $U_t$ must map $G_{a,b}$ onto $G_{a+t,b+t}$ and send an admissible norm into an admissible norm.

topology. One may assume that $G_{-\infty,+\infty}$ is just the union of all $G_{a,b}$, since anyway, only the local spaces $G_{a,b}$ will be used.

Given a sum system $\big((G_{a,b}),(U_t)\big)$, we may introduce (as explained in Sect. 5) Hilbert spaces $H_{a,b} = \operatorname{Exp}(G_{a,b})$ satisfying (under the usual identification)

$$H_{a,b} \otimes H_{b,c} = H_{a,c}$$

whenever $-\infty < a < b < c < +\infty$. Given $a < b < c < d$, $\psi_1 \in H_{a,b}$, $\psi_2 \in H_{b,c}$, $\psi_3 \in H_{c,d}$, we may calculate $\psi_1 \otimes \psi_2 \otimes \psi_3$ as $(\psi_1 \otimes \psi_2) \otimes \psi_3$ or $\psi_1 \otimes (\psi_2 \otimes \psi_3)$, which is the same (recall Lemma 3.5). The two-parameter families may be reduced to one-parameter families using $(U_t)$; namely, for $s,t \in (0,\infty)$,

$$\begin{aligned} G_{0,s+t} &= G_{0,s} \oplus U_s G_{0,t}\,, \\ H_{0,s+t} &= H_{0,s} \otimes \big(\operatorname{Exp}(U_s|_{G_{0,t}})\big) H_{0,t}\,, \end{aligned} \tag{7.2}$$

where $\big(\operatorname{Exp}(U_s|_{G_{0,t}})\big) : H_{0,t} \to H_{s,s+t}$ is the unitary operator corresponding to the FHS isomorphism $U_s|_{H_{0,t}} : H_{0,t} \to H_{s,s+t}$. The binary operation of tensor product, $(s,\psi_1),(t,\psi_2) \mapsto \big(s+t, \psi_1 \otimes (\operatorname{Exp}(U_s|_{G_{0,t}}))\psi_2\big)$ (for $\psi_1 \in H_{0,s}$, $\psi_2 \in H_{0,t}$) is associative. That is the algebraic part of the 'product system' structure defined by W. Arveson [4, Def. 1.4]. It is not the whole story, since some measurability in $s,t$ is needed. Namely, the disjoint union of spaces $H_{0,s}$ must be a standard Borel space, and the tensor product must be a measurable binary operation.

The disjoint union is the set of all pairs $(s,\psi)$ such that $s \in (0,\infty)$ and $\psi \in H_{0,s}$. We take some $T \in (0,\infty)$; it is enough to consider $s \in (0,T)$ rather than $(0,\infty)$. We have a Gaussian type space $(\Omega,\mathcal{F},\mathcal{P},G)$ such that $G/\operatorname{Const} = G_{0,T}$, and sub-$\sigma$-fields $\mathcal{F}_{a,b} \subset \mathcal{F}$ such that $L_2(\Omega,\mathcal{F}_{a,b},\mathcal{P}) = H_{a,b}$ whenever $(a,b) \subset (0,T)$.

**7.3. Lemma.** The map $(a,b) \mapsto G_{a,b}$ from the triangle $\{(a,b) : 0 \le a < b \le T\}$ to the Borel space $\mathbf{L}(G_{0,T})$ of subspaces of $G_{0,T}$ is Borel measurable.

*Proof.* It is enough to consider the case $0 \le a < t < b \le T$ for an arbitrary $t \in (0,T)$. The equality $G_{a,b} = G_{a,t} \oplus G_{t,b}$, in combination with Lemma 6.5, reduces the problem to measurability of $G_{a,t}$ in $a$, and $G_{t,b}$ in $b$. However, a *monotone* function is always measurable. □

So, $G_{a,b}$ is measurable in $(a,b)$. It follows by Lemma 6.17 that $\mathcal{F}_{a,b}$ is measurable in $(a,b)$. Proposition 6.14 gives us a Borel structure on the disjoint union of spaces $H_{a,b}$ (over all $a,b$ satisfying $0 \le a < b \le T$); it is compatible with linear operations and scalar product; and the map

$$\big(((a,b),\psi_1),((b,c),\psi_2)\big) \mapsto ((a,c),\psi_1 \otimes \psi_2)$$

is Borel measurable (here $\psi_1 \in H_{a,b}$, $\psi_2 \in H_{b,c}$, $0 \le a < b < c \le T$).

Given $a, b$ such that $0 \le a < b \le T$, we may treat the restriction $U_{b-a}|_{G_{0,a}}$ as an FHS isomorphism $G_{0,a} \to G_{b-a,b}$, therefore an element of the Borel space $\mathcal{I}(G_{0,T})$ introduced in Sect. 6.

**7.4. Lemma.** The map $(a,b) \mapsto U_{b-a}|_{G_{0,a}}$ from the triangle $\{(a,b) : 0 \le a < b \le T\}$ to $\mathcal{I}(G_{0,T})$ is Borel measurable.

*Proof.* By Lemma 7.3, $G_{0,a}$ is measurable in $a$. According to (6.1) there are $f_n(a) \in G_{0,a}$, measurable in $a$, such that every $G_{0,a}$ is spanned by $\{f_1(a), f_2(a), \dots\}$. Therefore the graph of $U_{b-a}|_{G_{0,a}}$ is spanned by pairs $\bigl(f_n(a), U_{b-a} f_n(a)\bigr)$. Each pair is measurable in $a$ and continuous in $b$ (recall 7.1(c)), therefore, measurable in $(a,b)$ (see [11, Th. 3.1.30]). $\square$

So, $U_{b-a}|_{G_{0,a}}$ is measurable in $(a,b)$. It follows by Proposition 6.20 that $\operatorname{Exp}(U_{b-a}|_{G_{0,a}})$ is also measurable in $(a,b)$. Lemma 6.16 shows that $\operatorname{Exp}(U_{b-a}|_{G_{0,a}})\psi$ is jointly measurable in $\psi \in H_{0,a}$ and $a, b$. It follows that $\psi_1 \otimes \bigl(\operatorname{Exp}(U_{b-a}|_{G_{0,a}})\bigr)\psi_2$ is jointly measurable in $a$, $b$, $\psi_1 \in H_{b-a}$, $\psi_2 \in H_a$, which proves the following result.

**7.5. Theorem.** If $\bigl((G_{a,b}), (U_t)\bigr)$ is a sum system, then Hilbert spaces $H_{a,b} = \operatorname{Exp}(G_{a,b})$ with the natural identification $H_{0,s+t} = H_{0,s} \otimes \bigl(\operatorname{Exp}(U_s|_{G_{0,t}})\bigr) H_{0,t}$ form a product system.

The product system may be called the exponential of the given sum system.

# 8 The invariant

An isomorphism of two product systems is defined [4, p. 6] as a family $(V_t)$ of linear isomorphisms $V_t : H'_{0,t} \to H''_{0,t}$ between corresponding Hilbert spaces that respects the two structures on the disjoint union of the Hilbert spaces, namely, the binary operation of tensor multiplication, and the Borel $\sigma$-field. Assuming that the two product systems are exponentials of two given sum systems $\bigl((G'_{a,b}), (U'_t)\bigr)$ and $\bigl((G''_{a,b}), (U''_t)\bigr)$, we may redefine equivalently an isomorphism as a two-parameter family $(V_{a,b})$ of unitary operators that satisfy

$$
\begin{gathered}
V_{a,b} : H'_{a,b} \to H''_{a,b} \text{ unitarily,} \\
V_{a,c} = V_{a,b} \otimes V_{b,c} \,, \\
\operatorname{Exp}(U''_t|_{G''_{a,b}}) V_{a,b} = V_{a+t,b+t} \operatorname{Exp}(U_t|_{G'_{a,b}}) \,,
\end{gathered}
$$

whenever $-\infty < a < b < c < +\infty$ and $-\infty < t < +\infty$ (as before, $H'_{a,b} = \operatorname{Exp}(G'_{a,b})$, $H''_{a,b} = \operatorname{Exp}(G''_{a,b})$), and respects the Borel structure on the disjoint union of Hilbert spaces.

For now $H_E$, as well as $G_E$ and $\mathcal{F}_E$, are defined only when $E$ is an interval, $E = (a,b)$. However, they may be defined for any elementary set $E$, that is, a union of a finite number of intervals. Given $-\infty < a < b < c < d < +\infty$, we have

$$\underbrace{H_{a,d}}_{H_{(a,d)}} = H_{a,b} \otimes H_{b,c} \otimes H_{c,d} = \underbrace{(H_{a,b} \otimes H_{c,d})}_{H_{(a,b)\cup(c,d)}} \otimes \underbrace{H_{b,c}}_{H_{(b,c)}} \, .$$

The same for any finite number of intervals. We get $H_{E_1 \cup E_2} = H_{E_1} \otimes H_{E_2}$ when $E_1 \cap E_2 = \emptyset$. Dealing with elementary sets we neglect boundary points, treating, say, $(a,b) \cup (b,c)$ as $(a,c)$. Also, $H_E = H_{E_1} \otimes \dots \otimes H_{E_n}$ whenever $E_1, \dots, E_n$ are pairwise disjoint and $E = E_1 \cup \dots \cup E_n$. Similarly, $G_E = G_{E_1} \oplus \dots \oplus G_{E_n}$ (in the FHS sense), and $\mathcal{F}_E = \mathcal{F}_{E_1} \otimes \dots \otimes \mathcal{F}_{E_n}$ (recall 3.2–3.5).

**8.1. Proposition.** Let two sum systems $\big((G'_{a,b}),(U'_t)\big)$ and $\big((G''_{a,b}),(U''_t)\big)$ be such that the corresponding product systems are isomorphic. Let $E_1, E_2, \dots \subset (0,1)$ be elementary sets. Then the following two conditions are equivalent.

(a) $\liminf G'_{(0,1)\setminus E_n} = G'_{(0,1)}$ and $\limsup G'_{E_n} = \{0\}$;

(b) $\liminf G''_{(0,1)\setminus E_n} = G''_{(0,1)}$ and $\limsup G''_{E_n} = \{0\}$.

(The FHS spaces are treated as subspaces of $G'_{(0,1)}$, $G''_{(0,1)}$ respectively.)

*Proof.* Theorem 5.3 allows us to reformulate the conditions in terms of density matrices in product systems, thus making explicit their invariance under isomorphisms. $\square$

# 9 Slightly coloured noises

Consider a scalar product of the form

$$\langle f, g \rangle = \iint f(s) g(t) B(s-t) \, ds dt \tag{9.1}$$

assuming that $B : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is continuous outside of the origin, and $B(-t) = B(t)$ for all $t \in (0,\infty)$, and $\int_0^1 |B(t)| \, dt < \infty$. The scalar product is well-defined whenever $f, g \in L_2(-M, M)$, $M \in (0,\infty)$. We assume that $B$ is positively definite in the sense that $\langle f, f \rangle \ge 0$ for all such $f$.

Denote by $G_{a,b}$ the completion of $L_2(a,b)$ w.r.t. the scalar product (9.1), then $G_{a,b}$ is a Hilbert space. We introduce operators $U_t$ by $(U_t f)(s) = f(s-t)$

for $f \in L_2(a,b)$ and extend $U_t$ by continuity to any $G_{a,b}$; thus, $U_t : G_{a,b} \to G_{a+t,b+t}$ is a unitary operator, and $U_s U_t = U_{s+t}$, and $\|U_t f - f\|_{G_{a,d}} \to 0$ for $t \to 0$, if $f \in G_{b,c}$ and $a < b < c < d$ (since it holds for the dense subset of continuous functions $f$).

In order to get a sum system (as defined by 7.1) we need to ensure that $G_{a,c} = G_{a,b} \oplus G_{b,c}$ in the FHS sense. The property will be verified for $B$ such that

$$\exists \varepsilon > 0 \ \forall t \in (0,\varepsilon) \ B(t) = \frac{1}{t \ln^\alpha (1/t)} \,; \tag{9.2}$$
$$\text{on } (0,\infty) \text{ the function } B(\cdot) \text{ is positive, decreasing and convex;}$$

here $\alpha \in (1,\infty)$ is a parameter. Such $B$ is positively definite, since it is an integral combination (with positive weights) of 'triangle' functions of the form $t \mapsto \max(0, a - |t|)$, and maybe a positive constant function.

We consider $G_{-T,0}$ and $G_{0,T}$ for an arbitrary $T \in (0,\infty)$. To this end we introduce $X_k \in G_{0,T}$ and $Y_k \in G_{-T,0}$ by

$$\begin{aligned} X_k(t) &= \mathbf{1}_{(0,T)}(t) \cdot \exp(2\pi i k t/T) \,, \\ Y_k(t) &= \mathbf{1}_{(-T,0)}(t) \cdot \exp(2\pi i k t/T) \end{aligned} \tag{9.3}$$

for $k \in \mathbb{Z}$; of course, $\mathbf{1}_{(a,b)}$ is the indicator of $(a,b)$. Clearly, $X_k$ span $G_{0,T}$, and $Y_k$ span $G_{-T,0}$. An elementary calculation gives

(9.4)
$$\begin{gathered} \langle X_k, X_k \rangle = \langle Y_k, Y_k \rangle = 2 \int_0^T (T-t) B(t) \cos(2\pi k t/T) \, dt \,, \\ \langle X_k, X_l \rangle = \langle Y_k, Y_l \rangle = - \frac{T}{\pi(k-l)} \int_0^T B(t) \big( \sin(2\pi k t/T) - \sin(2\pi l t/T) \big) \, dt \,, \\ \langle X_k, Y_k \rangle = \int_0^{2T} \min(t, 2T-t) B(t) \exp(2\pi i k t/T) \, dt \,, \\ \langle X_k, Y_l \rangle = -i \frac{T}{2\pi(k-l)} \int_0^{2T} B(t) \operatorname{sgn}(T-t) \big( \exp(2\pi i k t/T) - \exp(2\pi i l t/T) \big) \, dt \end{gathered}$$

for $k \ne l$. We want to estimate $\langle X_k, X_l \rangle$ and $\langle X_k, Y_l \rangle$ from above. These are increments of Fourier transforms, thus we want to differentiate these Fourier transforms. The singularity of $B$ at the origin contributes a term that decays slowly (near $\infty$) and is monotone. Jumps outside the origin (at $T$ and $2T$) contribute terms that decay much faster, but oscillate. After differentiation, these oscillating terms dominate the monotone term. However, we need Fourier transforms only on the lattice $(2\pi/T)\mathbb{Z}$, thus we have a freedom to

change the given functions without changing their Fourier transforms on the lattice. We'll use the freedom for eliminating the jumps.

Note that $\int e^{i\lambda t}(U_s f)(t)\,dt = e^{i\lambda s}\int e^{i\lambda t} f(t)\,dt$, therefore $\int e^{i\lambda t}(U_T f - f)(t)\,dt = 0$ for $\lambda \in (2\pi/T)\mathbb{Z}$. We use a piecewise linear $f$ for correcting $B$; namely, we define

$$
\begin{gathered}
b_1(t) = \begin{cases} B(t) - B(T)\cdot\frac{T-t}{T} & \text{for } 0 < t \le T,\\ B(T)\cdot\frac{2T-t}{T} & \text{for } T \le t \le 2T,\\ 0 & \text{for other } t, \end{cases}\\
\hat b_1(\lambda) = \int_0^\infty e^{i\lambda t} b_1(t)\,dt\,,
\end{gathered}
$$

then $b_1$ is continuous on $(0,\infty)$, and

$$
\langle X_k, X_l\rangle = \langle Y_k, Y_l\rangle = -\frac{T}{\pi(k-l)}\operatorname{Im}\bigl(\hat b_1(2\pi k/T) - \hat b_1(2\pi l/T)\bigr)
$$

for $k \ne l$. Similarly,

$$
\begin{gathered}
b_2(t) = \begin{cases} B(t) - B(t+T) - (B(T)-B(2T))\cdot\frac{T-t}{T} & \text{for } 0 < t \le T,\\ (B(T)-B(2T))\cdot\frac{2T-t}{T} & \text{for } T \le t \le 2T,\\ 0 & \text{for other } t, \end{cases}\\
\hat b_2(\lambda) = \int_0^\infty e^{i\lambda t} b_2(t)\,dt\,;\\
\langle X_k, Y_l\rangle = -i\frac{T}{2\pi(k-l)}\bigl(\hat b_2(2\pi k/T) - \hat b_2(2\pi l/T)\bigr) \quad \text{for } k \ne l\,.
\end{gathered}
$$

It is easy to check that both $b_1$ and $b_2$ satisfies the conditions of the following lemma, provided that $B$ satisfies (9.2).[33]

**9.5. Lemma.** Assume that $\alpha \in (1,\infty)$, and a function $b : (0,\infty) \to \mathbb{R}$ is continuous, and the difference $b(t) - \dfrac{\mathbf{1}_{(0,\varepsilon)}(t)}{t\ln^\alpha(1/t)}$ is of finite variation on $(0,\infty)$ for some (therefore, every) $\varepsilon \in (0,1)$, and $b(t) = 0$ for all $t$ large enough. Assume also that the function $t \mapsto tb(t)$ is absolutely continuous on $(0,\infty)$, and the difference $\bigl(tb(t)\bigr)' - \alpha\dfrac{\mathbf{1}_{(0,\varepsilon)}(t)}{t\ln^{\alpha+1}(1/t)}$ is of finite variation on $(0,\infty)$ for some (therefore, every) $\varepsilon \in (0,1)$. Then the function $\hat b(\lambda) = \int_0^\infty e^{i\lambda t} b(t)\,dt$ satisfies

$$
\begin{aligned}
\hat b(\lambda) &= \frac{1}{\alpha-1}\frac{1}{\ln^{\alpha-1}\lambda} + O\Bigl(\frac{1}{\ln^\alpha\lambda}\Bigr) &&\text{for } \lambda \to +\infty\,,\\
\frac{d}{d\lambda}\hat b(\lambda) &= -\frac{1}{\lambda\ln^\alpha\lambda} + O\Bigl(\frac{1}{\lambda\ln^{\alpha+1}\lambda}\Bigr) &&\text{for } \lambda \to +\infty\,.
\end{aligned}
$$

[33] Finite variation of $(tB(t))'$ on any $[\varepsilon, 1/\varepsilon]$ follows from increase of $(tB(t))' - 2B(t)$.

*Proof.* Choosing any $\varepsilon \in (0,1)$ we have for large $\lambda$

$$\hat{b}(\lambda) = \underbrace{\int_0^\infty e^{i\lambda t}\Big(b(t) - \frac{\mathbf{1}_{(0,\varepsilon)}(t)}{t\ln^\alpha(1/t)}\Big)\,dt}_{O(1/\lambda)} + \int_0^\varepsilon e^{i\lambda t}\frac{1}{t\ln^\alpha(1/t)}\,dt =$$

$$= \Big(\int_0^{1/\lambda} + \int_{1/\lambda}^\varepsilon\Big)e^{i\lambda t}\frac{1}{t\ln^\alpha(1/t)}\,dt + O(1/\lambda) =$$

$$= \int_0^{1/\lambda}\frac{1}{t\ln^\alpha(1/t)}\,dt + \int_0^{1/\lambda}\frac{e^{i\lambda t}-1}{t\ln^\alpha(1/t)}\,dt + \int_{1/\lambda}^\varepsilon e^{i\lambda t}\frac{1}{t\ln^\alpha(1/t)}\,dt + O(1/\lambda)\,;$$

$$\int_0^{1/\lambda}\frac{1}{\ln^\alpha(1/t)}\frac{dt}{t} = \frac{1}{\alpha-1}\frac{1}{\ln^{\alpha-1}\lambda}\,;$$

$$\Big|\int_0^{1/\lambda}\frac{e^{i\lambda t}-1}{t\ln^\alpha(1/t)}\,dt\Big| \le \int_0^{1/\lambda}\frac{\lambda t}{t\ln^\alpha(1/t)}\,dt = \lambda\int_0^{1/\lambda}\frac{dt}{\ln^\alpha(1/t)} \le \frac{1}{\ln^\alpha\lambda}\,;$$

$$\int_{1/\lambda}^\varepsilon\frac{e^{i\lambda t}}{t}\frac{1}{\ln^\alpha(1/t)}\,dt = \int_{1/\lambda}^\varepsilon\frac{1}{\ln^\alpha(1/t)}\,d\big(\mathrm{ci}(\lambda t) + i\,\mathrm{si}(\lambda t)\big)\,,$$

where $\mathrm{ci}(t) = -\displaystyle\int_t^\infty\frac{\cos u}{u}\,du$, $\mathrm{si}(t) = -\displaystyle\int_t^\infty\frac{\sin u}{u}\,du$. Taking into account that $\mathrm{ci}(t) = O(1/t)$ and $\mathrm{si}(t) = O(1/t)$, we get

$$\int_{1/\lambda}^\varepsilon e^{i\lambda t}\frac{1}{t\ln^\alpha(1/t)}\,dt = \underbrace{\frac{\mathrm{ci}(\lambda t) + i\,\mathrm{si}(\lambda t)}{\ln^\alpha(1/t)}\Big|_{1/\lambda}^\varepsilon}_{O(1/\ln^\alpha\lambda)} -$$

$$- \underbrace{\int_{1/\lambda}^\varepsilon\big(\mathrm{ci}(\lambda t) + i\,\mathrm{si}(\lambda t)\big)\cdot\frac{\alpha}{t\ln^{\alpha+1}(1/t)}\,dt}_{O(\int_{1/\lambda}^\varepsilon \frac{1}{\lambda t}\cdot\frac{dt}{t\ln^{\alpha+1}(1/t)})}\,;$$

$$\frac{1}{\lambda}\int_{1/\lambda}^\varepsilon\frac{dt}{t^2\ln^{\alpha+1}(1/t)} = \frac{1}{\lambda}\Big(\int_{1/\lambda}^{1/\sqrt{\lambda}} + \int_{1/\sqrt{\lambda}}^\varepsilon\Big)\frac{dt}{t^2\ln^{\alpha+1}(1/t)} \le$$

$$\le \frac{1}{\lambda}\frac{1}{\ln^{\alpha+1}\sqrt{\lambda}}\underbrace{\int_{1/\lambda}^\infty\frac{dt}{t^2}}_{\lambda} + \frac{1}{\lambda}\frac{1}{\ln^{\alpha+1}(1/\varepsilon)}\underbrace{\int_{1/\sqrt{\lambda}}^\infty\frac{dt}{t^2}}_{\sqrt{\lambda}} = O\Big(\frac{1}{\ln^{\alpha+1}\lambda}\Big)\,.$$

So,

$$\hat{b}(\lambda) = \frac{1}{\alpha-1}\frac{1}{\ln^{\alpha-1}\lambda} + O\Big(\frac{1}{\ln^\alpha\lambda}\Big)\quad\text{for } \lambda\to+\infty\,,$$

which is the first claim of the lemma. In order to prove the second claim we note that the only properties of the function $b(t)$ used till now are the finite variation of $b(t) - \frac{\mathbf{1}_{(0,\varepsilon)}(t)}{t \ln^\alpha(1/t)}$, and $b(t) = 0$ for large $t$. Therefore the same argument may be applied to the function $\frac{1}{\alpha}\bigl(tb(t)\bigr)'$ w.r.t. $\alpha + 1$:

$$\int_0^\infty e^{i\lambda t} \frac{1}{\alpha} \bigl(tb(t)\bigr)' \, dt = \frac{1}{\alpha} \frac{1}{\ln^\alpha \lambda} + O\Bigl(\frac{1}{\ln^{\alpha+1} \lambda}\Bigr) \, .$$

Hence

$$\begin{aligned} \frac{d}{d\lambda} \hat{b}(\lambda) &= \int_0^\infty e^{i\lambda t} itb(t) \, dt = \frac{1}{\lambda} \int_0^\infty tb(t) (e^{i\lambda t})' \, dt = \\ &= -\frac{1}{\lambda} \int_0^\infty \bigl(tb(t)\bigr)' e^{i\lambda t} \, dt = -\frac{1}{\lambda \ln^\alpha \lambda} + O\Bigl(\frac{1}{\lambda \ln^{\alpha+1} \lambda}\Bigr) \, . \end{aligned}$$

□

**9.6. Lemma.** Let $B$ satisfy (9.2) and $X_k, Y_k$ be defined by (9.3); then

$$\sum_{m,n:m \ne n} \frac{|\langle X_m, X_n \rangle|^2}{\|X_m\|^2 \|X_n\|^2} < \infty \, , \qquad \sum_{m,n} \frac{|\langle X_m, Y_n \rangle|^2}{\|X_m\|^2 \|Y_n\|^2} < \infty \, .$$

*Proof.* First, the function $tB(t)$ on $[0, T]$ is of finite variation, thus, using (9.4) and Lemma 9.5,

$$\begin{aligned} \langle X_k, X_k \rangle = 2T \underbrace{\int_0^T B(t) \cos(2\pi kt/T) \, dt}_{\hat{b}_1(2\pi k/T)} &- 2 \underbrace{\int_0^T tB(t) \cos(2\pi kt/T) \, dt}_{O(1/|k|)} \sim \\ &\sim 2T \cdot \frac{1}{\alpha - 1} \cdot \frac{1}{\ln^{\alpha-1}(2\pi |k|/T)} \sim \frac{2T}{\alpha - 1} \frac{1}{\ln^{\alpha-1} |k|} \, ; \end{aligned}$$

$$\frac{1}{\|X_k\|^2} = O\bigl(\ln^{\alpha-1} |k|\bigr)$$

for $k \to \pm\infty$.

Second, $\langle X_k, Y_k \rangle = \int_0^{2T} \min(t, 2T - t) B(t) \exp(2\pi ikt/T) = O(1/|k|)$, since $\min(t, 2T - t) B(t)$ is of finite variation on $[0, 2T]$. Hence $|\langle X_n, Y_n \rangle|^2 / (\|X_n\|^2 \|Y_n\|^2) = O\bigl(\frac{1}{n^2} \ln^{2\alpha-2} |n|\bigr)$ and

$$\sum_n \frac{|\langle X_n, Y_n \rangle|^2}{\|X_n\|^2 \|Y_n\|^2} < \infty \, .$$

Third, $|\langle X_n, X_{-n}\rangle| = O(1/|n|)$ and $|\langle X_n, Y_{-n}\rangle| = O(1/|n|)$, hence

$$\sum_n \frac{|\langle X_n, X_{-n}\rangle|^2}{\|X_n\|^2 \|X_{-n}\|^2} < \infty \,, \qquad \sum_n \frac{|\langle X_n, Y_{-n}\rangle|^2}{\|X_n\|^2 \|Y_{-n}\|^2} < \infty \,.$$

It is enough to prove that[34]

$$\sum_{m,n:\, m\pm n\ne 0} \frac{\ln^{\alpha-1}(|m|+2)\cdot \ln^{\alpha-1}(|n|+2)}{(m-n)^2} |\hat b(2\pi m/T) - \hat b(2\pi n/T)|^2 < \infty$$

for every function $b$ as in Lemma 9.5. Taking into account that $\hat b(-\lambda) = \overline{\hat b(\lambda)}$ we transform it into

$$\sum_{m,n:\, 0\le m<n} \ln^{\alpha-1}(m+2)\ln^{\alpha-1}(n+2)\cdot \biggl( \frac{|\hat b(2\pi m/T) - \hat b(2\pi n/T)|^2}{(n-m)^2} + \\ + \frac{|\overline{\hat b(2\pi m/T)} - \hat b(2\pi n/T)|^2}{(n+m)^2} \biggr) < \infty \,;$$

$$\sum_{m,n:\, 0\le m<n} \ln^{\alpha-1}(m+2)\ln^{\alpha-1}(n+2)\cdot \\ \cdot \biggl( \bigl(\operatorname{Re}\hat b(2\pi m/T) - \operatorname{Re}\hat b(2\pi n/T)\bigr)^2 \cdot \Bigl( \frac{1}{(n-m)^2} + \frac{1}{(n+m)^2} \Bigr) + \\ + \bigl(\operatorname{Im}\hat b(2\pi m/T) - \operatorname{Im}\hat b(2\pi n/T)\bigr)^2 \cdot \frac{1}{(n-m)^2} + \\ + \bigl(\operatorname{Im}\hat b(2\pi m/T) + \operatorname{Im}\hat b(2\pi n/T)\bigr)^2 \cdot \frac{1}{(n+m)^2} \biggr) < \infty \,;$$

it is enough to prove that

$$\sum_{m,n:\, 0\le m<n} \frac{\ln^{\alpha-1}(m+2)\ln^{\alpha-1}(n+2)}{(n-m)^2} |\hat b(2\pi m/T) - \hat b(2\pi n/T)|^2 < \infty \,, \tag{9.7}$$

(9.8)
$$\sum_{m,n:\, 0\le m<n} \frac{\ln^{\alpha-1}(m+2)\ln^{\alpha-1}(n+2)}{(n+m)^2} \bigl(\operatorname{Im}\hat b(2\pi m/T) + \operatorname{Im}\hat b(2\pi n/T)\bigr)^2 < \infty \,.$$

[34] You see, $\ln|n|$ is replaced with $\ln(|n|+2)$ in order to cover the small values, $n = -1, 0, 1$.

We treat separately two cases, $0 \le m < \sqrt{n}$ and $\sqrt{n} \le m < n$. The first case, $0 \le m < \sqrt{n}$, is simple; just using boundedness of $\hat{b}$ we have for (9.7) and (9.8) as well,

$$\sum_{m,n:\, 0 \le m < \sqrt{n}} \dots \le \text{const} \cdot \sum_{m,n: 0 \le m < \sqrt{n}} \frac{\ln^{2\alpha-2}(n+2)}{n^2} \le \\ \le \text{const} \cdot \sum_n \sqrt{n} \cdot \frac{\ln^{2\alpha-2}(n+2)}{n^2} < \infty \,.$$

We turn to the other case, $\sqrt{n} \le m < n$. Now $\ln(n+2) = O(\ln(m+2))$. Lemma 9.5 gives $\operatorname{Im} \hat{b}(\lambda) = O\big(\frac{1}{\ln^\alpha \lambda}\big)$, hence

$$\frac{\ln^{\alpha-1}(m+2)\ln^{\alpha-1}(n+2)}{(n+m)^2} \big(\operatorname{Im} \hat{b}(2\pi m/T) + \operatorname{Im} \hat{b}(2\pi n/T)\big)^2 = \\ = O\bigg( \frac{\ln^{\alpha-1}(m+2)\ln^{\alpha-1}(n+2)}{n^2} \Big( \frac{1}{\ln^{2\alpha}(m+2)} + \frac{1}{\ln^{2\alpha}(n+2)} \Big) \bigg) = \\ = O\bigg( \frac{\ln^{2\alpha-2}(n+2)}{n^2 \ln^{2\alpha}(m+2)} \bigg) = O\bigg( \frac{1}{n^2 \ln^2(n+2)} \bigg) \,;$$

summing over $m$ gives $O\big(\frac{1}{n \ln^2(n+2)}\big)$, a convergent series in $n$, which proves (9.8).

It remains to prove the most delicate case, (9.7) for $\sqrt{n} \le m < n$. Neglecting a finite number of terms, we get $m$ large enough for using the asymptotic relation of Lemma 9.5:

$$\hat{b}(2\pi m/T) - \hat{b}(2\pi n/T) = O\bigg( \int_{2\pi m/T}^{2\pi n/T} \frac{d\lambda}{\lambda \ln^\alpha \lambda} \bigg) = O\bigg( \frac{1}{\ln^{\alpha-1} m} - \frac{1}{\ln^{\alpha-1} n} \bigg) \,.$$

However, $(\ln m)^{-(\alpha-1)} - (\ln n)^{-(\alpha-1)} \le (\alpha-1)(\ln m)^{-\alpha}(\ln n - \ln m)$ and $(\ln m)^{-\alpha} \le (\frac12 \ln n)^{-\alpha}$, therefore

$$\hat{b}(2\pi m/T) - \hat{b}(2\pi n/T) = O\bigg( \frac{\ln n - \ln m}{\ln^\alpha n} \bigg) \,.$$

It is enough to prove that

$$\sum_{m,n:\, \sqrt{n} \le m < n} \frac{\ln^{\alpha-1} m \cdot \ln^{\alpha-1} n}{(n-m)^2} \bigg( \frac{\ln n - \ln m}{\ln^\alpha n} \bigg)^2 < \infty$$

or, equivalently,

$$\sum_n \frac{1}{n^2 \ln^2 n} \sum_{m:\, \sqrt{n} \le m < n} \bigg( \frac{\ln \frac{n}{m}}{1 - \frac{m}{n}} \bigg)^2 < \infty \,.$$

It remains to note that

$$\frac{1}{n} \sum_{m:\, \sqrt{n} \le m < n} \left( \frac{\ln \frac{n}{m}}{1 - \frac{m}{n}} \right)^2 \le \int_0^1 \left( \frac{\ln(1/u)}{1-u} \right)^2 du < \infty \,.$$

$\square$

**9.9. Proposition.** If $B$ satisfies (9.2) then $X_k$ (defined by (9.3)) are orthogonal w.r.t. some admissible norm on the FHS-space $G_{0,T}$.

*Proof.* Here is an equivalent formulation: there is an operator $A : l_2 \to G_{0,T}$ such that

$$A(c_1, c_2, \dots) = \sum_k \frac{c_k}{\|X_k\|} X_k \quad \text{for all } (c_1, c_2, \dots) \in l_2 \,,$$

and $A$ is an FHS-isomorphism, in other words, an equivalence operator in the sense of Feldman [7, Def. 1]. It means that $A$ is one-to-one onto, has a bounded inverse, and $\sqrt{A^*A} - I \in \mathcal{HS}$ (the Hilbert-Schmidt class of operators). The latter is equivalent to $A^*A - I \in \mathcal{HS}$, see [7, Lemma 1(b)].[35]

Matrix elements of $A^*A$ are $\dfrac{|\langle X_m, X_n \rangle|}{\|X_m\| \|X_n\|}$; Lemma 9.6 shows that $A^*A - I \in \mathcal{HS}$ and, of course, $A$ is bounded. It remains to prove that $A$ has a bounded inverse. The range of $A$ being evidently dense, we have to prove that $\|Ax\| \ge \varepsilon \|x\|$ for some $\varepsilon$, that is, 0 does not belong to the spectrum of $A^*A$. The spectrum accumulates to 1 only (since $A^*A - I \in \mathcal{HS}$); we have to prove that 0 is not an eigenvalue, that is,

$$\sum_k \frac{c_k}{\|X_k\|} X_k = 0 \quad \Longrightarrow \quad c_1 = c_2 = \dots = 0$$

for all $(c_1, c_2, \dots) \in l_2$. It is enough to prove that the following formula is a correct definition of (continuous) linear functionals $X^1, X^2, \dots$ on $G_{0,T}$:

$$X^k(g) = \frac{1}{T} \int_0^T g(t) \exp(-2\pi i k t / T) \, dt \quad \text{for } g \in G_{0,T} \,;$$

indeed, it will follow that

$$c_k = \|X_k\| \cdot X^k \left( \sum_l \frac{c_l}{\|X_l\|} X_l \right) .$$

[35] Though, his formulation of the lemma is incorrect, see the review 21#1546 in Mathematical Reviews.

The norm on $G_{0,T}$, defined in terms of $B(\cdot)$, uses $B(t)$ for $t \in [-T,T]$ only. Therefore we may assume that $B(t)$ vanishes outside of some bounded interval (and still satisfies (9.2)). For every $g \in L_2(0,T) \subset G_{0,T}$,

$$\|g\|^2_{G_{0,T}} = \int_{-\infty}^{+\infty} \hat{B}(\lambda) |\hat{g}(\lambda)|^2 \, d\lambda \, ;$$

here $\hat{B}$ is the Fourier transform (normalized as to be unitary) of $B$, and $\hat{g}$ — of $g$. The formula $(Zg)(\lambda) = \sqrt{\hat{B}(\lambda)} \hat{g}(\lambda)$ defines a linear isometric embedding $Z : G_{0,T} \to L_2(\mathbb{R})$ on the dense subset $L_2(0,T) \subset G_{0,T}$; we may extend $Z$ to the whole $G_{0,T}$ by continuity. Every $\varphi \in L_2(\mathbb{R})$ gives a linear functional on $G_{0,T}$, namely, $g \mapsto \int \sqrt{\hat{B}(\lambda)} \hat{g}(\lambda) \varphi(\lambda) \, d\lambda$. In order to get $X^k$, we take $\varphi$ such that $\sqrt{\hat{B}(\lambda)} \varphi(\lambda)$ is the Fourier transform of $(1/T) \exp(-2\pi i k t/T) \mathbf{1}_{(0,T)}$; it remains to verify that such $\varphi$ belongs to $L_2(\mathbb{R})$.

The function $(1/T) \exp(-2\pi i k t/T) \mathbf{1}_{(0,T)}$ is of finite variation; its Fourier transform is $O\big(1/\sqrt{1+\lambda^2}\big)$; it remains to check that

$$\int \frac{1}{(1+\lambda^2) \hat{B}(\lambda)} \, d\lambda < \infty \, .$$

However, the continuous function $\hat{B}$ never vanishes, and $\hat{B}(\lambda) \sim \dfrac{\text{const}}{\ln^{\alpha-1} |\lambda|}$ for $\lambda \to \pm\infty$ by Lemma 9.5. $\square$

**9.10. Proposition.** If $B$ satisfies (9.2) then $G_{-T,T} = G_{-T,0} \oplus G_{0,T}$ (in the FHS sense).

*Proof.* Vectors $X_k, Y_k$ (defined by (9.3)) are orthogonal w.r.t. some admissible norm on the FHS-space $G_{-T,T}$; the proof is quite similar to the proof of Proposition 9.9. $\square$

Combining Proposition 9.10 with elementary properties of spaces $G_{a,b}$ and operators $U_t$ mentioned in the beginning of the section, we get the following result.

**9.11. Theorem.** If $B$ satisfies (9.2) for some $\alpha \in (1,\infty)$, then $\big((G_{a,b}),(U_t)\big)$ is a sum system (as defined by 7.1).

# 10 The product systems are nonisomorphic and unitless

Sum systems given by Theorem 9.11 depend on the parameter $\alpha \in (1, \infty)$.[36] Their exponentials, given by Theorem 7.5, are product systems. Such product systems for different $\alpha$ are nonisomorphic, which will be shown using Proposition 8.1. Accordingly, we consider a sequence of elementary sets $E_n \subset (0,1)$, and we want to know, which sequences $(E_n)$ satisfy $\liminf G_{(0,1)\setminus E_n} = G_{(0,1)}$ and $\limsup G_{E_n} = \{0\}$; here $G_{E_n}$ (as well as $G_{(0,1)\setminus E_n}$) correspond (as explained in Sect. 8) to the sum system given by Theorem 9.11; recall that $\liminf$ was defined by 3.6, and $\limsup$ by 3.13.

**10.1. Lemma.** If $\operatorname{mes} E_n \to 0$ then $\liminf G_{(0,1)\setminus E_n} = G_{(0,1)}$. (Here $\operatorname{mes} E_n$ is Lebesgue measure of $E_n$.)

*Proof.* We have to represent an arbitrary vector $g \in G_{(0,1)}$ as $\lim g_n$ for some $g_n \in G_{(0,1)\setminus E_n}$. It is enough to consider a dense set of vectors $g$ (since $\liminf$ is always closed). Let $g \in L_2(0,1) \subset G_{(0,1)}$ and $g_n = g \cdot \mathbf{1}_{(0,1)\setminus E_n} \in G_{(0,1)\setminus E_n} \cap L_2(0,1)$. Clearly, $\operatorname{mes} E_n \to 0$ implies $g \cdot \mathbf{1}_{E_n} \to 0$ in $L_2(0,1)$, hence $g_n \to g$ in $L_2(0,1)$, therefore $g_n \to g$ in $G_{0,1}$. $\square$

**10.2. Lemma.** Assume that $B$ satisfies (9.2) for a given $\alpha \in (1,\infty)$, and $\operatorname{mes} E_n = o(1/\ln^{\alpha-1} n)$, and $E_n$ consists of (no more than) $n$ intervals. Then $\limsup G_{E_n} = \{0\}$.

*Proof.* Let $g_n \in G_{E_n}$, $\|g_n\| \le 1$; we have to prove that $g_n \to 0$ weakly. Introduce

$$h_n(t) = c \frac{n^2}{\ln^{(\alpha-1)/2} n} \cdot \mathbf{1}_{(-1/n^2, +1/n^2)}$$

where $c > 0$ is chosen such that for all $n$ large enough,

$$B - h_n * h_n \quad \text{is positively definite;}$$

in terms of Fourier transform it means that

$$\big(\hat h_n(\lambda)\big)^2 \le B(\lambda) \quad \text{for all } \lambda \in \mathbb{R}\,;$$

such $c$ exists due to the asymptotic relation $\hat B(\lambda) \sim \frac{\text{const}}{\ln^{\alpha-1}|\lambda|}$ for large $|\lambda|$ (recall 9.5). The positive definiteness means that

$$\|g * h_n\|_{L_2(\mathbb{R})} \le \|g\|_{G_{0,1}}$$

[36]That is, each such system corresponds to some $\alpha$. However, for a given $\alpha$ there is some freedom when choosing $B$ satisfying (9.2).

for all $g \in L_2(0,1) \subset G_{0,1}$ and then (extending the convolution operator by continuity) for all $g \in G_{0,1}$.

Denote by $E'_n$ the $(1/n^2)$-neighborhood of $E_n$, then $\operatorname{mes} E'_n \le (2n/n^2) + \operatorname{mes} E_n = o(1/\ln^{\alpha-1} n)$, and $g * h_n \in L_2(E'_n) \subset L_2(\mathbb{R})$.

We have to prove that $\varphi(g_n) \to 0$ for every linear functional $\varphi$ on $G_{0,1}$. We may restrict ourselves to a dense subset of functionals $\varphi$ (since $\|g_n\| \le 1$). In particular, we may consider only functionals $\varphi_k$ defined by

$$\varphi_k(g) = \int g(t) \exp(2\pi i k t)\, dt\,.$$

Taking into account that $\varphi_k(g * h_n) = \dfrac{2c}{\ln^{(\alpha-1)/2} n} \cdot \dfrac{\sin(2\pi k/n^2)}{2\pi k/n^2} \varphi_k(g)$ we see that the following would be enough:

$$\frac{1}{2c}\bigl(\ln^{(\alpha-1)/2} n\bigr)\varphi_k(g_n * h_n) \to 0 \quad \text{when } n \to \infty$$

for every $k$.

Recalling that $g_n * h_n \in L_2(E'_n)$ and $\|g_n * h_n\|_{L_2(\mathbb{R})} \le \|g_n\|_{G_{0,1}} \le 1$ we have

$$|\varphi_k(g_n * h_n)| \le \sqrt{\operatorname{mes} E'_n} \cdot \|g_n * h_n\|_{L_2(\mathbb{R})} = o\Bigl(\frac{1}{\ln^{(\alpha-1)/2} n}\Bigr)\,.$$

$\square$

**10.3. Lemma.** Assume that $B$ satisfies (9.2) for a given $\alpha \in (1,\infty)$. Then there exists a sequence of elementary sets $E_n \subset (0,1)$ such that $\operatorname{mes} E_n = O(1/\ln^{\alpha-1} n)$, and $E_n$ consists of $n$ intervals, and the relation $\limsup G_{E_n} = \{0\}$ is violated.

*Proof.* The construction is straightforward, just $n$ equidistant intervals of equal length:

$$E_n = \bigcup_{k=1}^{n} \Bigl( \frac{1}{n}\Bigl(k - \frac{1}{2} - \frac{1}{n \ln^{\alpha-1} n}\Bigr), \frac{1}{n}\Bigl(k - \frac{1}{2} + \frac{1}{n \ln^{\alpha-1} n}\Bigr)\Bigr)\,;$$

$$\operatorname{mes} E_n = \frac{2}{\ln^{\alpha-1} n}\,.$$

We have to find $g_n \in G_{E_n}$ and $g \in G_{0,1}$ such that

$$\sup_n \|g_n\|_{G_{0,1}} < \infty\,, \tag{10.4}$$

$$\limsup_n |\langle g_n, g\rangle_{G_{0,1}}| > 0\,. \tag{10.5}$$

Still, the construction is straightforward:

$$
\begin{aligned}
g_n &= (1/\operatorname{mes} E_n) \cdot \mathbf{1}_{E_n} = \frac{1}{2} \ln^{\alpha-1} n \cdot \mathbf{1}_{E_n} \,, \\
g &= \mathbf{1}_{(0,1)} \,;
\end{aligned}
$$

the proof of (10.4), (10.5) is more complicated. Fourier transform $f \mapsto \hat{f}$ will be used, $\hat{f}(\lambda) = \int e^{i\lambda t} f(t)\, dt$. Recall that

$$
2\pi \langle f_1, f_2 \rangle_{G_{0,1}} = \int_{-\infty}^{+\infty} \hat{B}(\lambda) \hat{f}_1(\lambda) \overline{\hat{f}_2(\lambda)} \, d\lambda
$$

for all $f_1, f_2 \in G_{0,1}$, and $\hat{B}(\lambda) \sim \frac{\text{const}}{\ln^{\alpha-1} |\lambda|}$ for large $|\lambda|$. An elementary calculation gives

$$
\begin{aligned}
\hat{g}(\lambda) &= \frac{e^{i\lambda} - 1}{i\lambda} \,; \\
\sum_{k=0}^{n-1} \exp(ik\lambda/n) &= \frac{1 - \exp(i\lambda)}{1 - \exp(i\lambda/n)} \,; \\
\hat{g}_n(\lambda) &= (\ln n)^{\alpha-1} \exp\Big( i \frac{\lambda}{2n} \Big) \frac{1 - \exp(i\lambda)}{1 - \exp(i\lambda/n)} \frac{1}{\lambda} \sin \frac{\lambda}{n \ln^{\alpha-1} n} \,; \\
|\hat{g}_n(\lambda)| &= (\ln n)^{\alpha-1} \frac{|\sin \frac{\lambda}{2}|}{|\sin \frac{\lambda}{2n}|} \frac{1}{|\lambda|} \Big| \sin \frac{\lambda}{n \ln^{\alpha-1} n} \Big| \,.
\end{aligned}
$$

Note that

$$
\frac{1}{2\pi n} \int_0^{2\pi n} \frac{\sin^2 \frac{\lambda}{2}}{\sin^2 \frac{\lambda}{2n}} \, d\lambda = \frac{1}{2\pi n} \int_0^{2\pi n} \Big| \sum_{k=0}^{n-1} \exp(ik\lambda/n) \Big|^2 d\lambda = n \,.
$$

We have

$$
2\pi \|g_n\|^2 = \int_{-\infty}^{+\infty} \hat{B}(\lambda) |\hat{g}_n(\lambda)|^2 \, d\lambda = 2 \bigg( \int_0^n + \int_n^\infty \bigg) \hat{B}(\lambda) |\hat{g}_n(\lambda)|^2 \, d\lambda \,.
$$

For $\lambda \in (0, n)$ we note that

$$
\begin{aligned}
&\hat{B}(\lambda) \le \hat{B}(0) \,, \\
&|\hat{g}_n(\lambda)| \le \frac{1}{n} \frac{|\sin \frac{\lambda}{2}|}{|\sin \frac{\lambda}{2n}|} \,, \\
&\int_0^n |\hat{g}_n(\lambda)|^2 \, d\lambda \le \frac{1}{n^2} \int_0^n \frac{\sin^2 \frac{\lambda}{2}}{\sin^2 \frac{\lambda}{2n}} \, d\lambda \le 2\pi \,,
\end{aligned}
$$

therefore

$$\int_0^n \hat{B}(\lambda)|\hat{g}_n(\lambda)|^2\,d\lambda \le 2\pi\hat{B}(0) \quad \text{for all } n\,.$$

For $\lambda \in (n,\infty)$ we note that $\hat{B}(\lambda) \le \frac{\text{const}}{\ln^{\alpha-1} n}$ and change the scale, introducing $u = \dfrac{\lambda}{n\ln^{\alpha-1} n}$:

$$\int_n^\infty \hat{B}(\lambda)|\hat{g}_n(\lambda)|^2\,d\lambda \le \frac{\text{const}}{\ln^{\alpha-1} n}(\ln n)^{2(\alpha-1)}\int_n^\infty \frac{\sin^2\frac{\lambda}{2}}{\sin^2\frac{\lambda}{2n}}\frac{1}{\lambda^2}\sin^2\frac{\lambda}{n\ln^{\alpha-1} n}\,d\lambda \le$$
$$\le \frac{\text{const}}{n}\int_0^\infty \frac{\sin^2(n\omega_n u)}{\sin^2(\omega_n u)}\frac{\sin^2 u}{u^2}\,du\,,$$

where $\omega_n = \frac{1}{2}\ln^{\alpha-1} n \to \infty$. On each period, $u \in \left(\frac{\pi}{\omega_n}k, \frac{\pi}{\omega_n}(k+1)\right)$, we substitute $\frac{\sin^2 u}{u^2}$ by its maximal value:

$$\int_0^\infty \frac{\sin^2(n\omega_n u)}{\sin^2(\omega_n u)}\frac{\sin^2 u}{u^2}\,du \le$$
$$\le (1+o(1))\left(\int_0^\infty \frac{\sin^2 u}{u^2}\,du\right)\underbrace{\left(\frac{\omega_n}{\pi}\int_0^{\pi/\omega_n}\frac{\sin^2(n\omega_n u)}{\sin^2(\omega_n u)}\,du\right)}_{=n}\,.$$

Hence $\int_n^\infty \hat{B}(\lambda)|\hat{g}_n(\lambda)|^2\,d\lambda = O(1)$. So, (10.4) is verified.

Further,

$$2\pi\langle g_n, g\rangle = \int_{-\infty}^{+\infty}\hat{B}(\lambda)\hat{g}_n(\lambda)\overline{\hat{g}(\lambda)}\,d\lambda = \left(\int_{|\lambda|<M} + \int_{|\lambda|>M}\right)\hat{B}(\lambda)\hat{g}_n(\lambda)\overline{\hat{g}(\lambda)}\,d\lambda$$

for any $M \in (0,\infty)$. It is easy to see that

$$\hat{g}_n(\lambda) \xrightarrow[n\to\infty]{} \hat{g}(\lambda) \quad \text{uniformly in } \lambda \in [-M,M]\,,$$

which implies

$$\int_{|\lambda|<M}\hat{B}(\lambda)\hat{g}_n(\lambda)\overline{\hat{g}(\lambda)}\,d\lambda \to \int_{|\lambda|<M}\hat{B}(\lambda)|\hat{g}(\lambda)|^2\,d\lambda \quad \text{for } n\to\infty\,.$$

On the other hand,

$$\left|\int_{|\lambda|>M}\hat{B}(\lambda)\hat{g}_n(\lambda)\overline{\hat{g}(\lambda)}\,d\lambda\right| \le$$
$$\le \underbrace{\left(\int_{|\lambda|>M}\hat{B}(\lambda)|\hat{g}_n(\lambda)|^2\,d\lambda\right)^{1/2}}_{\le\sup_n\|g_n\|}\underbrace{\left(\int_{|\lambda|>M}\hat{B}(\lambda)|\hat{g}(\lambda)|^2\,d\lambda\right)^{1/2}}_{\to 0 \text{ for } M\to\infty}\,,$$

it tends to 0 uniformly in $n$, when $M \to \infty$. Choose $M$ and $\varepsilon > 0$ such that

$$\sup_n \biggl| \int_{|\lambda|>M} \hat{B}(\lambda) \hat{g}_n(\lambda) \overline{\hat{g}(\lambda)} \, d\lambda \biggr| \le \frac{1}{2} \varepsilon \quad \text{and} \quad \int_{|\lambda|<M} \hat{B}(\lambda) |\hat{g}(\lambda)|^2 \, d\lambda \ge \varepsilon \,,$$

then

$$\liminf_n \operatorname{Re} \langle g_n, g \rangle \ge \frac{1}{2} \frac{1}{2\pi} \varepsilon \,,$$

which implies (10.5). $\square$

**10.6. Theorem.** Let[37] $B', B''$ satisfy (9.2) for some $\alpha', \alpha''$ respectively, $\alpha', \alpha'' \in (1, \infty)$, $\alpha' \ne \alpha''$. Then the corresponding product systems are nonisomorphic.

*Proof.* Suppose that $\alpha' < \alpha''$. Lemma 10.3 gives elementary sets $E_n \subset (0,1)$ such that $\operatorname{mes} E_n = O(1/\ln^{\alpha''-1} n)$, and $E_n$ consists of $n$ intervals, and the relation $\limsup G''_{E_n} = \{0\}$ is violated. Taking into account that $O(1/\ln^{\alpha''-1} n) = o(1/\ln^{\alpha'-1} n)$ we get $\limsup G'_{E_n} = \{0\}$ by Lemma 10.2. Also, $\liminf G'_{(0,1)\setminus E_n} = G'_{(0,1)}$ and $\liminf G''_{(0,1)\setminus E_n} = G''_{(0,1)}$ by Lemma 10.1. So, $E_n$ satisfy 8.1(a) but violate 8.1(b). By Proposition 8.1 the product systems are nonisomorphic. $\square$

**10.7. Lemma.** Assume that $B$ satisfies (9.2) for a given $\alpha \in (1, \infty)$, and

$$\begin{aligned}
&E_n = \bigcup_{k=1}^n \biggl( \frac{1}{n} \Bigl( k - \frac{1}{2} - \frac{\operatorname{mes} E_n}{2n} \Bigr), \frac{1}{n} \Bigl( k - \frac{1}{2} + \frac{\operatorname{mes} E_n}{2n} \Bigr) \biggr) \,, \\
&(\ln n)^{\alpha-1} \operatorname{mes} E_n \to \infty \quad \text{for } n \to \infty \,, \\
&g_n = (1/\operatorname{mes} E_n) \cdot \mathbf{1}_{E_n} \,, \\
&g = \mathbf{1}_{(0,1)} \,.
\end{aligned}$$

Then $\| g_n - g \|_{G_{0,1}} \to 0$ for $n \to \infty$.

*Proof.* Similarly to the proof of Lemma 10.3 we have for any $M \in (0, \infty)$

$$\hat{g}_n(\lambda) \xrightarrow[n\to\infty]{} \hat{g}(\lambda) \quad \text{uniformly on } \lambda \in [-M, M] \,.$$

This time, however, the following compactness property holds:

$$\int_{|\lambda|>M} \hat{B}(\lambda) |\hat{g}_n(\lambda)|^2 \, d\lambda \xrightarrow[M\to\infty]{} 0 \quad \text{uniformly in } n \,,$$

[37]Here $B'$ does not mean the derivative of $B$, sorry.

which ensures $\|g_n - g\| \to 0$. In order to prove the compactness property we estimate integrals similarly to the proof of 10.3. We have

$$|\hat{g}_n(\lambda)| = \frac{2}{\operatorname{mes} E_n} \frac{|\sin\frac{\lambda}{2}|}{|\sin\frac{\lambda}{2n}|} \frac{1}{|\lambda|} \Big| \sin \frac{\lambda \operatorname{mes} E_n}{2n} \Big| .$$

For $\lambda \in (M, n)$ we note that[38]

$$\begin{aligned}
&\hat{B}(\lambda) \le \frac{\text{const}}{\ln^{\alpha-1} M} \,, \\
&|\hat{g}_n(\lambda)| \le \frac{1}{n} \frac{|\sin\frac{\lambda}{2}|}{|\sin\frac{\lambda}{2n}|} \,, \qquad \text{(the same as in 10.3)} \\
&\int_M^n \hat{B}(\lambda) |\hat{g}_n(\lambda)|^2 \, d\lambda \le \frac{\text{const}}{\ln^{\alpha-1} M} \int_0^n |\hat{g}_n(\lambda)|^2 \, d\lambda \le \frac{\text{const}}{\ln^{\alpha-1} M} \cdot 2\pi \xrightarrow[M\to\infty]{} 0 \,.
\end{aligned}$$

For $\lambda \in (n, \infty)$, introducing $u = \dfrac{\operatorname{mes} E_n}{2n} \lambda$ and $\omega_n = 1/\operatorname{mes} E_n \to \infty$, we have

$$\begin{aligned}
\int_n^\infty \hat{B}(\lambda) |\hat{g}_n(\lambda)|^2 \, d\lambda &\le \frac{\text{const}}{\ln^{\alpha-1} n} \Big( \frac{2}{\operatorname{mes} E_n} \Big)^2 \int_n^\infty \frac{\sin^2\frac{\lambda}{2}}{\sin^2\frac{\lambda}{2n}} \frac{1}{\lambda^2} \sin^2 \frac{\lambda \operatorname{mes} E_n}{2n} \, d\lambda \le \\
&\le \underbrace{\frac{\text{const}}{\ln^{\alpha-1} n} \frac{2}{\operatorname{mes} E_n}}_{\to 0} \cdot \frac{1}{n} \cdot \underbrace{\int_0^\infty \frac{\sin^2(n\omega_n u)}{\sin^2(\omega_n u)} \frac{\sin^2 u}{u^2} \, du}_{O(n)} \xrightarrow[n\to\infty]{} 0 \,.
\end{aligned}$$

$\square$

**10.8. Proposition.** If $B$ satisfies (9.2) for some $\alpha \in (1, \infty)$ then the corresponding product system is unitless.

*Proof.* Assume the contrary, then there exist $\psi_{a,b} \in H_{a,b} = \operatorname{Exp} G_{a,b}$ such that $\|\psi_{a,b}\| = 1$ and $\psi_{a,b} \otimes \psi_{b,c} = \psi_{a,c}$ whenever $a < b < c$. We'll use $\psi_{a,b}$ for $0 \le a < b \le 1$ only. Introduce a Gaussian type space $(\Omega, \mathcal{F}, \mathcal{P}, G)$ such that $G_{0,1} = G/\operatorname{Const}$; we have $H_{a,b} = L_2(\Omega, \mathcal{F}_{a,b}, \mathcal{P}|_{\mathcal{F}_{a,b}})$ where $\mathcal{F}_{a,b} \subset \mathcal{F}$ is the $\sigma$-field generated by the Gaussian subspace $G_{a,b} \subset G_{0,1}$. Consider measures $\mu_{a,b} = |\psi_{a,b}|^2$; as was explained in Sect. 1, $\mu_{a,b}$ is a probability measure on the $\sigma$-field $\mathcal{F}_{a,b}$ such that

$$\frac{\mu_{a,b}}{P|_{\mathcal{F}_{a,b}}} = \bigg| \frac{\psi_{a,b}}{\sqrt{P|_{\mathcal{F}_{a,b}}}} \bigg|^2 \quad \text{for all } P \in \mathcal{P} \,.$$

[38] Assuming that $M$ is large enough.

The relation $\psi_{a,b} \otimes \psi_{b,c} = \psi_{a,c}$ implies

$$\mu_{a,b} \otimes \mu_{b,c} = \mu_{a,c} \, .$$

In other words, $\sigma$-fields $\mathcal{F}_{a,b}$ and $\mathcal{F}_{b,c}$ are independent w.r.t. the measure $\mu = \mu_{0,1}$ on $\mathcal{F}$, and $\mu_{a,b}$ is just the restriction of $\mu$ to $\mathcal{F}_{a,b}$.[39] Moreover, $\sigma$-fields $\mathcal{F}_{t_0,t_1}, \mathcal{F}_{t_1,t_2}, \dots, \mathcal{F}_{t_{n-1},t_n}$ are $\mu$-independent whenever $0 \le t_0 < t_1 < \dots < t_n \le 1$. Every elementary set $E \subset (0,1)$ determines its sub-$\sigma$-field $\mathcal{F}_E \subset \mathcal{F}$; as explained in Sect. 8,

$$E = (t_0,t_1) \cup (t_2,t_3) \cup \dots \cup (t_{2n}, t_{2n+1}) \quad \Longrightarrow \quad \mathcal{F}_E = \mathcal{F}_{t_0,t_1} \otimes \dots \otimes \mathcal{F}_{t_{2n},t_{2n+1}}$$

whenever $0 \le t_0 < t_1 < \dots < t_{2n+1} \le 1$. Note that $\mathcal{F}_{E_1}, \mathcal{F}_{E_2}$ are $\mu$-independent whenever $E_1 \cap E_2 = \emptyset$.

Consider elementary sets

$$\begin{aligned} E_n &= \Big( 0, \frac{1}{2n} \Big) \cup \Big( \frac{2}{2n}, \frac{3}{2n} \Big) \cup \dots \cup \Big( \frac{2n-2}{2n}, \frac{2n-1}{2n} \Big) \, , \\ \operatorname{mes} E_n &= \frac{1}{2} \, , \end{aligned}$$

and vectors

$$\begin{aligned} g_n &= 2 \cdot \mathbf{1}_{E_n} \in G_{E_n} \subset G_{0,1} \, , \\ g &= \mathbf{1}_{(0,1)} \in G_{0,1} \, , \\ h_n &= 2 \cdot \mathbf{1}_{(0,1) \setminus E_n} \in G_{(0,1) \setminus E_n} \subset G_{0,1} \, . \end{aligned}$$

Lemma 10.7 shows[40] that $g_n \to g$ and $h_n \to g$ in $G_{0,1}$. Though, $G_{0,1}$ is not a subspace but a quotient space $G / \operatorname{Const}$ of $G$; anyway, we may choose elements of $G$, denoted again by $g_n, g, h_n$, such that

$$\begin{aligned} & g_n \in L_0(\Omega, \mathcal{F}_{E_n}, \mathcal{P}) \, , \\ & g \in L_0(\Omega, \mathcal{F}, \mathcal{P}) \, , \\ & h_n \in L_0(\Omega, \mathcal{F}_{(0,1) \setminus E_n}, \mathcal{P}) \, , \\ & g_n \to g \quad \text{and} \quad h_n \to g \quad \text{in } L_0(\Omega, \mathcal{F}, \mathcal{P}) \, . \end{aligned}$$

The natural map[41] $L_0(\Omega, \mathcal{F}, \mathcal{P}) \to L_0(\Omega, \mathcal{F}, \mu)$ allows us to treat $g_n, g, h_n$ as elements of $L_0(\Omega, \mathcal{F}, \mu)$. Now they are random variables; $g_n \to g$ and $h_n \to g$ in probability. On the other hand, for every $n$, the two random

[39] Clearly, $\mu$ is a probability measure on $(\Omega, \mathcal{F})$, absolutely continuous w.r.t. any $P \in \mathcal{P}$. However, $P$ need not be absolutely continuous w.r.t. $\mu$; that is, $\mu$ need not belong to $\mathcal{P}$.

[40] Shifting the set $E_n$ of Lemma 10.7 does not invalidate the lemma.

[41] Generally, non-invertible, since $\mu$ need not belong to $\mathcal{P}$.

variables $g_n, h_n$ are independent (since $\mathcal{F}_{E_n}$ and $\mathcal{F}_{(0,1)\setminus E_n}$ are $\mu$-independent). It follows that $g$ is independent of itself, that is, $g$ is constant $\mu$-almost sure.[42]

Consider a Gaussian measure $\gamma \in \mathcal{P}_G$. Though, $g$ need not be constant $\gamma$-almost sure; however, $g$ must be constant on a set of positive probability w.r.t. $\gamma$. On the other hand, the distribution of $g$ w.r.t. $\gamma$ is normal (Gaussian); it cannot have an atom unless it is degenerate, which means that $\|g\|_{G_{0,1}}$ must vanish. However, it does not vanish, which is evident when using Fourier transform. A contradiction. $\square$

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[42]Proof: let $\varphi : \mathbb{R} \to \mathbb{R}$ be a continuous and bounded function, then $\int \varphi(g_n(\omega))\varphi(h_n(\omega))\,\mu(d\omega) = \big(\int \varphi(g_n(\omega))\,\mu(d\omega)\big) \cdot \big(\int \varphi(h_n(\omega))\,\mu(d\omega)\big)$ due to independence. The limit for $n \to \infty$ gives $\int \big(\varphi(g(\omega))\big)^2\,\mu(d\omega) = \big(\int \varphi(g(\omega))\,\mu(d\omega)\big)^2$, which is impossible unless $g = \text{const}$.

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